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Today's topics

Reminder: The final homework assignment is due this week Friday, not next Monday.

1 Substitution rule

Briggs–Cochran–Gillett §5.5, pp. 388–398

At the end of the last class, we integrated $\cos 2x$, which brings us to a useful strategy: substitution. If we were to guess at the value of the indefinite integral $\int \cos 2x dx$, perhaps we would start with $\int \cos x\, dx = \sin x + C$. We might incorrectly conclude that the indefinite integral of $\cos 2x$ is $\sin 2x + C$, but differentiation by the Chain Rule would reveal that

$$\frac{d}{dx}(\sin 2x + C) = 2\cos 2x \neq \cos 2x.$$

But now it's pretty clear that we were just off by a factor of 2 and that

$$\int \cos 2x\, dx = \frac{1}{2}\sin 2x + C.$$

While this works here, this sort of trial-and-error approach is not practical for more complicated integrals, so we introduce the more systematic strategy of *substitution*, which we illustrate in the example of $\int \cos 2x dx$.

We first make the change of variable $u = 2x$. Then taking d's on both sides, this gives $du = 2dx$. We now rewrite our given integral:

$$\int \cos 2x\, dx = \int \cos u \frac{du}{2} = \frac{1}{2}\int \cos u\, du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin 2x + C.$$

We formalize this process as follows:

Theorem 1 (Substitution rule for indefinite integrals)**.** *Let $u = g(x)$, where g' is continuous on an interval, and let f be continuous on the corresponding range of g. On that interval,*

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

In practice, we apply this theorem as follows:

Substitution Rule (Change of variables)

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u.
4. Rewrite the result in terms of x using $u = g(x)$.

Disclaimer: Not all integrals yield to the Substitution Rule!

Example 2. Find the following indefinite integrals.

1. $\int xe^{x^2}\,dx$
2. $\int x^3(x^4+16)^6\,dx$
3. $\int \frac{3}{1+25y^2}\,dy$
4. $\int \frac{x}{\sqrt{x-4}}\,dx$
5. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

Solution. In each case, we make a substitution that simplifies the integral:

1. Using $u = x^2$, we have $du = 2x\,dx$, so
$$\int xe^{x^2}\,dx = \int \frac{1}{2}e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

2. Using $u = x^4 + 16$, we have $du = 4x^3\,dx$, so
$$\int x^3(x^4+16)^6\,dx = \int \frac{1}{4}u^6\,du = \frac{1}{28}u^7 + C = \frac{1}{28}(x^4+16)^7 + C.$$

3. Using $u = 5y$, we have $du = 5dy$, so
$$\int \frac{3}{1+25y^2}\,dy = \int \frac{3}{1+u^2}\,\frac{du}{5} = \frac{3}{5}\int \frac{1}{1+u^2}\,du = \frac{3}{5}\tan^{-1}(u) + C = \frac{3}{5}\tan^{-1}(5y) + C.$$

4. Using $u = x - 4$, we have $du = dx$, so
$$\begin{aligned}\int \frac{x}{\sqrt{x-4}}\,dx = \int \frac{u+4}{\sqrt{u}}\,du = \int (u^{1/2} + 4u^{-1/2})\,du &= \frac{2}{3}u^{3/2} + 8u^{1/2} + C \\ &= \frac{2}{3}(x-4)^{3/2} + 8(x-4)^{1/2} + C.\end{aligned}$$

5. Using $u = e^x + e^{-x}$, we have $du = (e^x - e^{-x})\,dx$, so
$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln(e^x + e^{-x}) + C.$$

   (Note that we can get rid of the absolute value because $e^x + e^{-x}$ is always positive.)

Next time, we'll see how to apply integration by substitution to definite integrals.