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Today's topics

Reminder: The final homework assignment is due this week Friday, not next Monday.

1 Substitution rule for definite integrals

Briggs–Cochran–Gillett §5.5, pp. 388–398

Theorem 1 (Substitution rule for definite integrals)**.** *Let $u = g(x)$, where g' is continuous on $[a, b]$ and let f be continuous on the range of g. Then*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example 2. Compute the following integrals.

1. $\displaystyle\int_0^2 \frac{2x}{(x^2+1)^2}\,dx$ (hint: try $u = x^2 + 1$)

2. $\displaystyle\int_0^4 \frac{p}{\sqrt{9+p^2}}\,dp$ (hint: try $u = 9 + p^2$ or $u = \sqrt{9+p^2}$; either will work)

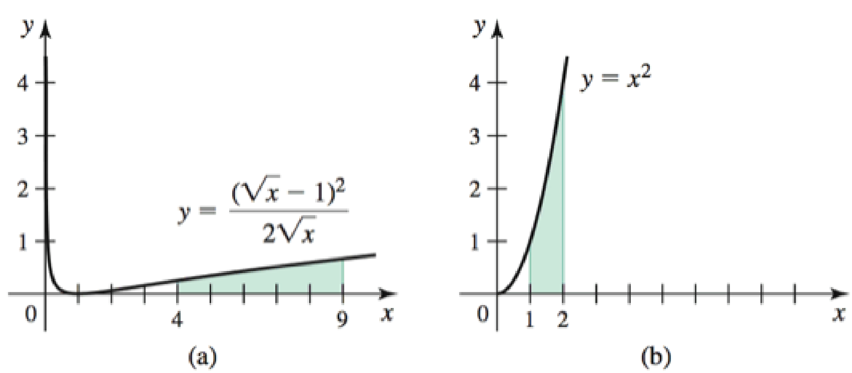
3. $\displaystyle\int x\cos^2(x^2)\,dx$ (hint: try $u = x^2$ and use $\cos^2(u) = \frac{1}{2}\cos(2u) + \frac{1}{2}$)

4. $\int (x^{3/2}+8)^5 \sqrt{x}\, dx$ (hint: try $u = x^{3/2}+8$)

5. $\int_0^{\pi/4} e^{\sin^2 x} \sin(2x)\, dx$ (hint: try $u = \sin^2 x$ and use $\sin(2x) = 2\sin x \cos x$)

Example 3. Find the area of the region bounded by the graph of $f(x) = \dfrac{x}{\sqrt{x^2-9}}$ and the x-axis between $x = 4$ and $x = 5$.

Example 4. The area of the shaded region under the curve $y = \dfrac{(\sqrt{x}-1)^2}{2\sqrt{x}}$ on the interval $[4, 9]$ in (a) equals the area of the shaded region under the curve $y = x^2$ on the interval $[1, 2]$ in (b). Without computing areas, explain why.



1.1 Further examples

Example 5. Evaluate the following integrals:

1. $\displaystyle\int \left(5f^3(x) + 7f^2(x) + f(x)\right)\ f'(x)\,dx$

   (hint: using $u = f(x)$ and $du = f'(x)dx$, we get the integral of a polynomial)

2. $\displaystyle\int_0^1 f'(x)f''(x)\,dx$, where $f'(0) = 3$ and $f'(1) = 2$.

   Solution. Using the substitution $u = f'(x)$ and $du = f''(x)dx$, we get

$$\int_0^1 f'(x)f''(x)\,dx = \int_{f'(0)}^{f'(1)} u\,du = \int_3^2 u\,du = \frac{1}{2}u^2\Big|_3^2 = \frac{1}{2}\cdot 2^2 - \frac{1}{2}\cdot 3^2 = \frac{-5}{2}.$$

Example 6. What is $\displaystyle\int \tan(x)\,dx$?

Solution. Using the substitution $u = \cos x$ and $du = -\sin x\,dx$, we have

$$\int \tan(x)\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-1}{u}\,du = -\ln|u| + C = -\ln|\cos x| + C.$$

Note that $-\ln|\cos x| = \ln|\cos x|^{-1} = \ln|\sec x|$, so you will sometimes see this written as $\int \tan(x)\,dx = \ln|\sec x| + C$.

Here is a table of integrals you should know:

$\frac{d}{du}F(u) = f(u)$	$\int f(u)\,du = F(u) + C$
$\frac{d}{du}u^{n+1} = (n+1)u^n$	$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
$\frac{d}{du}\ln u = \frac{1}{u}$	$\int \frac{1}{u}\,du = \ln\lvert u\rvert + C$
$\frac{d}{du}\sin u = \cos u$	$\int \cos u\,du = \sin u + C$
$\frac{d}{du}\cos u = -\sin u$	$\int \sin u\,du = -\cos u + C$
$\frac{d}{du}\tan u = \sec^2 u$	$\int \sec^2 u\,du = \tan u + C$
$\frac{d}{du}\sec u = \sec u \tan u$	$\int \sec u \tan u\,du = \sec u + C$
$\frac{d}{du}e^u = e^u$	$\int e^u\,du = e^u + C$
$\frac{d}{du}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}$	$\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}u + C$
$\frac{d}{du}\tan^{-1}u = \frac{1}{1+u^2}$	$\int \frac{1}{1+u^2}\,du = \tan^{-1}u + C$
$\frac{d}{du}\sec^{-1}u = \frac{1}{\lvert u\rvert\sqrt{u^2-1}}$	$\int \frac{1}{\lvert u\rvert\sqrt{u^2-1}}\,du = \sec^{-1}u + C$