1 Systems of Linear Equations

1.1 Two equations in two unknowns

This vast and varied mathematical body of theorems, algorithms and numerical observations that we have come to call applied linear algebra evolved by an enduring preoccupation with the theory and practice of the solution of large systems of linear equations.

Look first at the modest, yet instructive, *square* 2×2 system

$$\begin{aligned} ax + by &= f \\ a'x + b'y &= f' \end{aligned} \tag{1.1}$$

of two equations in two unknowns, x and y. Numbers a, b, a', b' are the *coefficients* of the system and numbers f, f' are the *right-hand sides* of the system. We shall try to write x and y separately in terms of the given coefficients and right-hand sides.

If in system (1.1) it happens that $b = 0$ and $a' = 0$, then the system *decouples* inasmuch as x is readily obtained from the first equation alone, and y from the second equation alone. To have a *coupled* or *simultaneous* system we assume that no coefficient in system (1.1) is zero. To solve the coupled system we *isolate* x in the first equation, rewriting the equation as

$$x = \frac{1}{a}(f - by) \tag{1.2}$$

then we *substitute* this x, written now in terms of y, into the second equation to have

$$\frac{a'}{a}(f - by) + b'y = f'. \tag{1.3}$$

Unknown x is thereby *eliminated* from the second equation, leaving in it only unknown y. Unknown y is computed from eq. (1.3) in terms of the coefficients and right-hand sides and is *backsubstituted* into eq. (1.2) to similarly obtain x.

When all this is done, x and y are algebraically expressed in terms of the coefficients a, a', b, b', and right-hand sides f, f' as

$$x = \frac{1}{\Delta}(fb' - f'b), \ y = \frac{1}{\Delta}(f'a - fa'), \ \Delta = ab' - a'b \tag{1.4}$$

where characteristic number Δ is called the *determinant* of the system.

We readily verify that in spite of the assumption on the non vanishing of coefficient a, undertaken in deriving eq. (1.4), the equation stays valid even for $a = 0$.

Algebraic solution (1.4) to system (1.1) demonstrates that a *unique solution to the system of linear equations exists if and only if the determinant of its coefficients is nonzero*, $\Delta \neq 0$. For a vanishing determinant, $\Delta = 0$, system (1.1) possesses an *infinity* of solutions if in addition both $fb' - f'b = 0$ and $f'a - fa' = 0$, due to the 0/0 ambiguity; or the system is *insoluble* (not an equation) if $fb' - f'b \neq 0$ or $f'a - fa' \neq 0$, due to the 1/0 impossibility. A soluble system of linear equations is said to be *consistent* while the insoluble system is said to be *inconsistent*.

It is interesting that only one single function, the determinant Δ, of all the many coefficients of the system, suffices to determine the condition of solution uniqueness.

A system with zero right-hand sides, $f = f' = 0$, is called *homogeneous*. A homogeneous system is always soluble, the *trivial solution* $x = y = 0$ being an immutable solution thereof.

Example. Consider the the three systems of linear equations

$$\begin{aligned} 2x - 3y &= 5 \qquad & x - y &= -1 \qquad & x - y &= 1 \\ -x + 2y &= -3 \qquad & -x + y &= 1 \qquad & -x + y &= 2. \end{aligned} \tag{1.5}$$

Elimination of x from each second equation of systems (1.5), the way we algebraically did it in eqs. (1.2) and (1.3), leaves us with the three correspondingly modified systems

$$\begin{aligned} 2x - 3y &= 5 \qquad & x - y &= -1 \qquad & x - y &= 1 \\ 1y &= -1 \qquad & 0y &= 0 \qquad & 0y &= 3. \end{aligned} \tag{1.6}$$

The second equation ,$1y = -1$, of the first of the above three systems yields the unequivocal $y = -1$, and consequently, $x = 1$. These are the only x and y values that satisfy, at once, both equations of the first system in eq. (1.5). The second equation, $0y = 0$, of the second modified system above implies an arbitrary y, as any number times zero results in zero, and consequently $x = y - 1$ for any y. Any pair of numbers $x = y - 1,\ y = y$, for arbitrary y satisfies both equations of the second system in eq. (1.5). The second equation, $0y = 3$, of the third modified system is an impossibility. Never can a number times zero equal 3, and there is no value for y to solve the third system. The third system is inconsistent and hides a contradiction. To reveal the contradiction, that is not too subtle here, we rewrite the system as $x - y = 1,\ x - y = -2$, which is obviously impossible for any x and y.

We have just made the important discovery that systems of linear equations, in analogy with the single, linear, one-unknown equation $ax = f$, may possess a unique solution, many solutions, or no solution at all, except that the uniqueness condition $a \neq 0$ for the single equation is replaced here by the comprehensive $\Delta \neq 0$ condition for the system. Before this chapter ends we will have reached the general conclusion that *only square systems of linear equations, those that actually have the same number of equations as unknowns, can have a unique solution*, and hence our keen interest in such systems.

The apparently naive procedure practiced above to write down the solution of system (1.1) is essentially the *Gauss elimination* algorithm, which with only modest technical refinement and systematization, is a mainstay of computational linear algebra. This chapter is concerned with the organizational detail and theoretical considerations of the Gauss elimination algorithm for the systematic, safe, and efficient solution of large systems of algebraic equations.

Exercises.

1.1.1. Solve the three systems of equations

$$\begin{aligned} -x + 3y &= 4 \\ 2x - y &= -3, \end{aligned} \qquad \begin{aligned} -x + 2y &= 1 \\ 2x - 4y &= -1, \end{aligned} \qquad \begin{aligned} 2x - y &= 3 \\ -6x + 3y &= -9. \end{aligned}$$

1.1.2. For what values of f, f' does system

$$x - y = f$$

$$-x + y = f'$$

possess multiple solutions?

1.1.3. For what values of coefficient α does system

$$x - y = 1$$

$$-x + \alpha y = -1$$

have a unique solution? Can it happen that for some α the system is insoluble?

1.1.4. For what values of f, f', f'' is system

$$2x - y + 3z = f$$

$$3x - 4y + 5z = f'$$

$$x - 3y + 2z = f''$$

solved by $x = -1 - 7t,\ y = 2 + t,\ z = 1 + 5t$, for *arbitrary* t?

1.1.5. For what values of $f,\ f',\ f''$ is system

$$2x - y + 3z = f$$

$$3x - 4y + 5z = f'$$

$$x - 3y + 2z = f''$$

solved by $x = -7t,\ y = t,\ z = 5t$ for arbitrary t?

1.1.6. For what values of α and β does the homogeneous system

$$\alpha x - y = 0$$

$$x + \beta y = 0$$

have a unique solution?

1.2 A word on determinants

They are a notational marvel of mathematics, an expanded way of writing of the difference between two products that exposes an expressional pattern leading to obvious generalizations and uncaping a fountain of mathematical fascination.

Writing

$$ab' - a'b = \begin{vmatrix} a & b \\ a' & b' \end{vmatrix} \tag{1.7}$$

appears ungainly and space consuming, but in this mode of writing, or in this *determinant notation*, the solution to system (1.1) is revealingly written as

$$x = \frac{1}{\Delta}\begin{vmatrix} f & b \\ f' & b' \end{vmatrix}, y = \frac{1}{\Delta}\begin{vmatrix} a & f \\ a' & f' \end{vmatrix}, \ \Delta = \begin{vmatrix} a & b \\ a' & b' \end{vmatrix} \tag{1.8}$$

bidding, at a glance, formal generalizations to larger or *higher order* systems.

Guided by the transparent pattern of eq. (1.8) in its relationship to the structure of system (1.1) we venture to formally write the solution of the 3×3 linear system

$$\begin{aligned} ax + by + cz &= f \\ a'x + b'y + c'z &= f' \\ a''x + b''y + c''z &= f'' \end{aligned} \tag{1.9}$$

of three equations in three unknowns as

$$x = \frac{1}{\Delta}\begin{vmatrix} f & b & c \\ f' & b' & c' \\ f'' & b'' & c'' \end{vmatrix} \quad y = \frac{1}{\Delta}\begin{vmatrix} a & f & c \\ a' & f' & c' \\ a'' & f'' & c'' \end{vmatrix} \quad z = \frac{1}{\Delta}\begin{vmatrix} a & b & f \\ a' & b' & f' \\ a'' & b'' & f'' \end{vmatrix}. \tag{1.10}$$

But what is it that we have so boldly written? To see what the larger Δ means we proceed to recursively construct the solution of the 3×3 system from the known solution of the 2×2 system. For this we write the last two equations of system (1.9) as

$$\begin{aligned} b'y + c'z &= f' - a'x \\ b''y + c''z &= f'' - a''x \end{aligned} \tag{1.11}$$

and solve them for y and z by means of eq. (1.8) as

$$y = \frac{1}{\Delta_1}\left(\begin{vmatrix} f' & c' \\ f'' & c'' \end{vmatrix} - x\begin{vmatrix} a' & c' \\ a'' & c'' \end{vmatrix}\right),\ z = \frac{1}{\Delta_1}\left(\begin{vmatrix} b' & f' \\ b'' & f'' \end{vmatrix} + x\begin{vmatrix} a' & b' \\ a'' & b'' \end{vmatrix}\right)$$
$$\Delta_1 = \begin{vmatrix} b' & c' \\ b'' & c'' \end{vmatrix}. \tag{1.12}$$

Substitution of y and z, as written above in terms of x, into the first equation of system (1.9) leaves us with one equation in the one unknown x, and upon solving this equation we discover the recursive formula

$$\Delta = \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} = a\begin{vmatrix} b' & c' \\ b'' & c'' \end{vmatrix} - b\begin{vmatrix} a' & c' \\ a'' & c'' \end{vmatrix} + c\begin{vmatrix} a' & b' \\ a'' & b'' \end{vmatrix}. \tag{1.13}$$

for expressing the (3×3) determinant in terms of three (2×2) determinants.

In general, a determinant of *order* n (an $n \times n$ determinant) consists of n *rows* or n *columns* and is recursively defined in terms of lower-order determinants as

$$\Delta = a_1\Delta_1 - a_2\Delta_2 + a_3\Delta_3 - \cdots + (-1)^{n+1}a_n\Delta_n \tag{1.14}$$

where $a_1, a_2, \ldots, a_n$ are the entries of the first row, and where Δ_j denotes the $(n-1)$th order determinant obtained by deleting the first row and jth column of Δ.

Generalization of determinant formulas (1.8) and (1.10) to the solution of larger systems of equations is known as *Cramer's rule*. We perceive a pleasant regularity in Cramer's rule, and herein lies its main apparent advantage—it is easy to remember. Once the algebraic system of linear equations, say (1.9), is given through its coefficients and right-hand sides, the determinants needed for its solution in form (1.10) are readily written down through a cyclic replacement of the first, second and third columns of determinant Δ by the right-hand sides f, f', f'', and then the easily remembered recursive expansion rule (1.14) for determinants is applied. But yet we are far from being done with square systems of linear equations.

Cramer's rule is good for paper and pencil solution of very small systems. For the solution of larger systems of linear equations we rely on programmable computers which do not have memory predicaments of a psychological nature; they are never forgetful. It is not the *formational simplicity* of the solution algorithm that is computationally important but rather its *efficiency* and *stability*, and Cramer's rule is notoriously inefficient. Formula (1.10) contains numerous redundant arithmetical operations.

An inkling as to why Cramer's rule is so exorbitant is gained by looking at the *diagonal* system

$$\begin{array}{rcl} a_1x & = & f_1 \\ b_2y & = & f_2 \\ c_3z & = & f_3 \end{array} \tag{1.15}$$

that, with no rule or formula, is immediately solved with 3 divisions. By Cramer's rule

$$x = \frac{1}{\Delta}\begin{vmatrix} f_1 & 0 & 0 \\ f_2 & b_2 & 0 \\ f_3 & 0 & c_3 \end{vmatrix} = \frac{f_1b_2c_3}{a_1b_2c_3}, \quad y = \frac{1}{\Delta}\begin{vmatrix} a_1 & f_1 & 0 \\ 0 & f_2 & 0 \\ 0 & f_3 & c_3 \end{vmatrix} = \frac{a_1f_2c_3}{a_1b_2c_3}$$
$$z = \frac{1}{\Delta}\begin{vmatrix} a_1 & 0 & f_1 \\ 0 & b_2 & f_2 \\ 0 & 0 & f_3 \end{vmatrix} = \frac{a_1b_2f_3}{a_1b_2c_3} \tag{1.16}$$

where

$$\Delta = \begin{vmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{vmatrix} = a_1b_2c_3 \tag{1.17}$$

is the determinant of the system. Even if our hypothetical determinant evaluation computer program, written to carry out expansion rule (1.14) is clever enough to avoid zero arithmetical operations, still eight multiplications and three divisions are needed to evaluate x, y, z in eq. (1.16). In case of an $n \times n$ diagonal system, Cramer's rule requires nearly n^2 multiplications and n divisions, as opposed to a mere n divisions in the direct solution.

Arithmetical redundancy in the determinant formula results from the fact that one single number, the determinant $\Delta = a_1b_2c_3$, is called upon to express the condition that the list of numbers a_1, b_2, c_3 does not contain a zero.

The general situation is much worse. The number of multiplications involved in evaluating an $n \times n$ determinant is a rough measure of the arithmetical work spent on it. From the recursive formula (1.14) we know that if the arithmetical work needed to evaluate an $(n-1)\times(n-1)$ determinant is w_{n-1}, then the work needed to evaluate an $n \times n$ determinant is $w_n = nw_{n-1} + n$, implying that, approximately, $w_n = n!$, and meaning that even on the swiftest of computers, working the recursive ladder to evaluate a 20×20 determinant should take forever. Actually, by successive eliminations and back substitutions the computer does not require more than a tiny fraction of a second to solve a linear system of 20 equations in 20 unknowns. Careful analysis shows that the time it takes to solve an $n \times n$ linear system is not a *factorial* function of n but rather *polynomial*—it being proportional to n^3.

Cramer's rule for the numerical solution of a square system of equations is aesthetically appealing but appallingly inefficient. Admittedly, with Cramer's rule we are able to write the solution *algebraically* in terms of the coefficients and right-hand sides, but we are rarely interested in this.

There are other numerical woes associated with the computation of determinants. As hinted by $\Delta = a_1 b_2 c_3$ for the diagonal system, the determinants of large systems can either become so huge as to cause Cramer's rule to break down in the computer *overflow* $x = \infty/\infty$, or so small as to cause the *underflow* $x = 0/0$. Also, since expansion of a determinant includes many terms that may be much larger or much smaller in magnitude than the coefficients, the result of their summation can be seriously affected by *rounding* errors in *floating-point* computations. In fact, from

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 b_2 c_3 + a_2 c_1 b_3 + a_3 b_1 c_2 - a_1 c_2 b_3 - a_2 c_3 b_1 - a_3 c_1 b_2 \tag{1.18}$$

we derive the total differential

$$\begin{aligned} d\Delta = da_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - da_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + da_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ - db_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + db_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - db_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} \\ + dc_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - dc_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + dc_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \end{aligned} \tag{1.19}$$

for the linearized change $d\Delta$ in Δ resulting from the changes da_1, da_2, da_3; db_1, db_2, db_3; dc_1, dc_2, dc_3; in the corresponding coefficients, and the nine terms in eq. (1.19) can add up to a substantial $d\Delta$. Computation of Δ by the formula in eq. (1.18) is inherently inaccurate: small changes in the numerical values of the coefficients can cause a significant change in the value of the determinant. For example

$$\begin{vmatrix} 5 & 4 & 4 \\ 8 & 5 & 6 \\ 7 & 6 & 5 \end{vmatrix} = 5, \qquad \text{while} \qquad \begin{vmatrix} 4.9 & 4.1 & 4.1 \\ 8.1 & 4.9 & 5.9 \\ 7.1 & 6.1 & 4.9 \end{vmatrix} = 10.26. \tag{1.20}$$

We shall say, therefore, nothing further on Cramer's rule beyond what we may directly verify for the 3×3 system. Nevertheless, we do make a distinction between the determinant of the coefficients of a system of linear equations — which is a *scalar function* of the coefficients of considerable theoretical interest and importance — and the determinant *notation* that draws its allure from the special way in which a multilinear algebraic expression is written in a two-dimensional format of rows and columns. We shall return to discuss the determinant of the coefficients of a general square system of linear equations in chapter 2.

As for the determinant notation, because of its expressional advantage in various formulas, because it provides for witty exercises and interesting theorems, and because of its historical entrenchment, we shall not entirely ignore it, but shall relegate further discussion of it to the exercises.

Exercises.

1.2.1. Write the 2×3 system

$$\begin{matrix} ax + by + cz = f \\ a'x + b'y + c'z = f' \end{matrix} \quad \text{as} \quad \begin{matrix} ax + by = f - cz \\ a'x + b'y = f' - c'z \end{matrix}$$

and argue, by considering the possible values of

$$\Delta = \begin{vmatrix} a & b \\ a' & b' \end{vmatrix}$$

that the system cannot have a unique solution.

1.2.2. Expand a 2×3 system into a 3×3 system by adding to it a $0 = 0$ equation so as to have

$$\begin{matrix} 0x + 0y + 0z = 0 \\ ax + by + cz = f \\ a'x + b'y + c'z = f' \end{matrix} \ , \quad \text{with the determinant} \quad \Delta = \begin{vmatrix} 0 & 0 & 0 \\ a & b & c \\ a' & b' & c' \end{vmatrix}.$$

Evaluate Δ and argue that the 2×3 system may never have a unique solution.

1.2.3. Consider the 3×2 and 3×3 systems

$$\begin{matrix} ax + by = f \\ a'x + b'y = f' \\ a''x + b''y = f'' \end{matrix} \qquad \begin{matrix} ax + by + cz = g \\ a'x + b'y + c'z = g' \\ a''x + b''y + c''z = g'' \end{matrix} \ .$$

If the second system has a unique solution, then so does the first, provided that $f = g - cz,\ f' = g' - c'z,\ f'' = g'' - c''z$.

Argue that the the 2×3 system has a unique solution if c, c', c'' can be found so that

$$\Delta = \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} \neq 0,$$

and that the condition for this is that at least one of the three determinants

$$\Delta_1 = \begin{vmatrix} a & b \\ a' & b' \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} a & b \\ a'' & b'' \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a' & b' \\ a'' & b'' \end{vmatrix}$$

does not vanish.

1.2.4. Solve both

$$\begin{vmatrix} 1 & -2 & 1 \\ 1 & x & 2 \\ 1 & x & -1 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} x & -1 & 1 \\ 1 & x & 1 \\ 1 & -1 & 0 \end{vmatrix} = -2.$$

1.2.5. Solve

$$\begin{vmatrix} 1 & -1 & 2 \\ 3 & x & 1 \\ 1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -2 & x \end{vmatrix}.$$

1.2.6. Show that

$$\frac{d}{dt}\begin{vmatrix} 1 & t^2 \\ 1 & t \end{vmatrix} = \begin{vmatrix} 1 & 2t \\ 1 & 1 \end{vmatrix}.$$

Show more generally that

$$\frac{d}{dt}\begin{vmatrix} f_1 & g_1 \\ f_2 & g_2 \end{vmatrix} = \begin{vmatrix} \dot{f}_1 & g_1 \\ \dot{f}_2 & g_2 \end{vmatrix} + \begin{vmatrix} f_1 & \dot{g}_1 \\ f_2 & \dot{g}_2 \end{vmatrix},$$

where $\dot{f} = df/dx$. Extend this formula to a 3×3 determinant.

1.2.7. Fix x in

$$\begin{vmatrix} a & a & a \\ a & b & b \\ a & b & c \end{vmatrix} = a(x-a)(c-x).$$

1.2.8. Show that

$$\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0 \ \text{ but } \ \begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix} = (af - bc + cd)^2.$$

1.2.9. Show that

$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix} = 0$$

if $x + y + z = 0$.

1.2.10. Show that

$$\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = 0$$

if $x = a$, or $x = -2a$.

1.2.11. Solve

$$\begin{vmatrix} x & 1 & 0 \\ 1 & x & 0 \\ 0 & 1 & x \end{vmatrix} = 0.$$

1.2.12. Solve

$$\begin{vmatrix} x & 0 & 0 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = 0.$$

1.2.13. Show that

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} = \frac{1}{abc} \begin{vmatrix} 1 & 1 & 1 \\ a'bc & b'ca & c'ab \\ a''bc & b''ca & c''ab \end{vmatrix}, \quad abc \neq 0.$$

1.2.14. Prove the equality

$$\begin{vmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{vmatrix} = (a_1 - a_2)(a_2 - a_3)(a_3 - a_1)$$

for this *Vandermonde* determinant.

1.2.15. Prove that if $b_1 \neq 0$, then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \frac{1}{b_1}\left(\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} - \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \right).$$

1.2.16. Show that if

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad \text{and} \quad \Delta' = \begin{vmatrix} a_1' & b_1' \\ a_2' & b_2' \end{vmatrix},$$

then

$$\begin{vmatrix} a_1 + a_1{}' & b_1 + b_1{}' \\ a_2 + a_2{}' & b_2 + b_2{}' \end{vmatrix} = \Delta + \Delta' + \begin{vmatrix} a_1 & b_1{}' \\ a_2 & b_2{}' \end{vmatrix} + \begin{vmatrix} a_1{}' & b_1 \\ a_2{}' & b_2 \end{vmatrix}.$$

1.2.17. Prove that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} a_1{}' & b_1{}' \\ a_2{}' & b_2{}' \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ 0 & 0 & a_1{}' & b_1{}' \\ 0 & 0 & a_2{}' & b_2{}' \end{vmatrix} = \begin{vmatrix} a_1a_1{}' + b_1a_2{}' & a_1b_1{}' + b_1b_2{}' \\ a_2a_1{}' + b_2a_2{}' & a_2b_1{}' + b_2b_2{}' \end{vmatrix}.$$

1.2.18. Consider

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \ \Delta' = \begin{vmatrix} f_1 & b_1 \\ f_2 & b_2 \end{vmatrix}, \ \Delta'' = \begin{vmatrix} a_1 & f_1 \\ a_2 & f_2 \end{vmatrix}.$$

If $\Delta = 0$, can f_1 and f_2 be found so that $\Delta' = 0$ but $\Delta'' \neq 0$?

1.2.19. Consider

$$\Delta = \begin{vmatrix} 1 & 5 & 6 \\ -2 & 7 & 5 \\ 3 & -1 & 2 \end{vmatrix}, \Delta' = \begin{vmatrix} f_1 & 5 & 6 \\ f_2 & 7 & 5 \\ f_3 & -1 & 2 \end{vmatrix}, \Delta'' = \begin{vmatrix} 1 & f_1 & 6 \\ -2 & f_2 & 5 \\ 3 & f_3 & 2 \end{vmatrix}, \Delta''' = \begin{vmatrix} 1 & 5 & f_1 \\ -2 & 7 & f_2 \\ 3 & -1 & f_3 \end{vmatrix}.$$

Show that $\Delta = 0$ and find f_1, f_2, f_3 so that $\Delta' = \Delta'' = 0$, $\Delta''' \neq 0$.

1.2.20. Prove that

$$\begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{vmatrix} = a_1b_2c_3.$$

1.2.21. Prove that

$$\begin{vmatrix} pa_1 & qb_1 & rc_1 \\ pa_2 & qb_2 & rc_2 \\ pa_3 & qb_3 & rc_3 \end{vmatrix} = pqr \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

1.2.22. Prove that

$$\begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix} = 0.$$

1.2.23. Prove that

$$\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0,$$

or that the determinant is zero if any two of its columns (rows) are equal.

1.2.24. Prove that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

or that the determinant does not change its value by a corresponding interchange of rows by columns.

1.2.25. Prove that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix},$$

or that the determinant changes sign upon an interchange of any two of its columns (rows).

1.2.26. Prove that

$$\begin{vmatrix} a_1 & pa_1 + b_1 & c_1 \\ a_2 & pa_2 + b_2 & c_2 \\ a_3 & pa_3 + b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

or that the determinant does not change if to the entries of any column the corresponding entries of another column times p are added. Use exercise 1.2.24 to argue that what is true for the columns is true for the rows.

1.2.27. Show by subtraction of rows that

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

1.2.28. Show that

$$\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{vmatrix} = 1.$$

1.2.29. Compute both

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \quad \text{and} \quad \Delta' = \begin{vmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 10 & 11 \end{vmatrix}.$$

1.2.30. Compute both

$$\Delta = \begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 4^2 & 5^2 & 6^2 \\ 7^2 & 8^2 & 9^2 \end{vmatrix} \quad \text{and} \quad \Delta' = \begin{vmatrix} 2^2 & 3^2 & 4^2 \\ 5^2 & 6^2 & 7^2 \\ 8^2 & 9^2 & 10^2 \end{vmatrix}.$$

1.2.31. A 2×2 determinant with ± 1 entries, a *Hadamard* determinant, can have a maximum value of 2. This happens when in

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1,\ \ a_1b_2 = 1,\ \ a_2b_1 = -1$$

or $a_2 = -1/b_1,\ \ b_2 = 1/a_1$, and $a_1 = \pm 1,\ \ b_1 = \pm 1$,

$$\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix} = 2.$$

According to eq. (1.18) the highest value a 3×3 Hadamard determinant can possibly reach is 6. Does a 3×3 Hadamard determinant exist that actually has the value 6? Maybe 5?

1.2.32. Evaluate

$$H = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix}.$$

1.2.33. Let

$$\Delta_n = \begin{vmatrix} a & c & 0 & 0 & 0 \\ b & a & c & 0 & 0 \\ 0 & b & a & c & 0 \\ 0 & 0 & b & a & c \\ 0 & 0 & 0 & b & a \end{vmatrix}$$

be a determinant of order n. Use eq. (1.14) to prove that

$$\Delta_n = a\Delta_{n-1} - bc\Delta_{n-2}.$$

1.2.34. For Δ_n in problem 1.2.33, show that if $a = 2, b = c = -1$, then $\Delta_n = 2\Delta_{n-1} - \Delta_{n-2}$, and that since $\Delta_1 = 2$ and $\Delta_2 = 3$, $\Delta_n = n + 1$.

1.2.35. For Δ_n in problem 1.2.33, show that if $a = 3, b = c = -1$, then $\Delta_n = 3\Delta_{n-1} - \Delta_{n-2}$. Show that $\Delta_n = z^n$ satisfies $\Delta_n = 3\Delta_{n-1} - \Delta_{n-2}$ if $z^n = 3z^{n-1} - z^{n-2}$, or $z^2 - 3z + 1 = 0$. Show that if z_1 and z_2 are the two roots of the quadratic equation, then $\Delta_n = \alpha_1 z_1^n + \alpha_2 z_2^n$ for any α_1 and α_2. Fix α_1 and α_2 so that $\Delta_1 = 3$ and $\Delta_2 = 8$, and verify that $\Delta_3 = 21, \Delta_7 = 987$ and $\Delta_{11} = 46368$.

Notice that in case $a = 1, b = -1, c = 1$ the recursive formula $\Delta_n = \Delta_{n-1} + \Delta_{n-2}$ generates Δ_n in the Fibonacci sequence 1,2,3,5,8,13,21,... .

1.2.36. For Δ_n in problem 1.2.33, show that if $a = 1, b = c = -1$, then $\Delta_n = \Delta_{n-1} - \Delta_{n-2}$, and $\Delta_{3n-1} = 0$.

1.3 Notational frugalities

Dealing with large systems of linear equations, we stand in pressing need of a manageable notational convention to write them down economically and systematically. The notation should permit varying degrees of conciseness and abstraction to allow for selective and compact exhibition of only those features of the system that are essential to a specific argument. For the sake of typographical clarity and brevity we want to do away with any notational irrelevancies.

As a first step in this direction we avoid relying on the order of the English alphabet, with its limited supply of symbols for denoting unknowns, and prefer to write the general system of m equations in n unknowns (the $m \times n$ system) with only three *subscripted* letters:

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= f_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= f_2 \\ \vdots \qquad\qquad\qquad\qquad & \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &= f_m. \end{aligned} \tag{1.21}$$

In this notation A_{ij} is the coefficient of the jth unknown x_j in the ith equation, and f_i is the right-hand side of the ith equation. A system of linear equations with any number of unknowns and equations is written in this notation in an orderly and systematic manner.

But the writing is still greatly redundant, for if the first term in each equation is always with x_1, and correspondingly the jth term is always with x_j, then there is no need to repeatedly write the unknowns in each equation. They can be filled in by location, as long as tabs are kept on their order of appearance. Moreover, we may do away with the multitude of $+$'s and $=$'s, and sparely write the algebraic system of linear equations in the tabular form

$$\begin{bmatrix} A_{11} & A_{12} & & A_{1n} \\ A_{21} & A_{22} & & A_{2n} \\ & & & \\ A_{m1} & A_{m2} & & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \\ x_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \\ f_m \end{bmatrix}. \tag{1.22}$$

The two-dimensional table or array that holds the A_{ij} coefficients of the system is called the *coefficient matrix* of the system. For system (1.21) the coefficient matrix has m rows and n columns. Entry A_{ij} of matrix A is located in the ith row and jth column of the matrix. In a square system, entry A_{ii} is a *diagonal* coefficient of the system, while entry A_{ij} $i \neq j$ is an *off-diagonal* coefficient of the system.

Soon, in chapter 2, we will endow the coefficient matrix with a mathematical life of its own.

To understand why it is important to keep a record of unknowns $x_1, x_2, \ldots, x_n$, as we did in the vertical list in eq. (1.22), suppose that for some reason we wish to interchange in each equation the location of the x_1 term with that of the x_2 term. Then the rewritten system reads

$$\begin{bmatrix} A_{12} & A_{11} & & A_{1n} \\ A_{22} & A_{21} & & A_{2n} \\ & & & \\ A_{m2} & A_{m1} & & A_{mn} \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ x_3 \\ \\ x_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \\ f_m \end{bmatrix} \tag{1.23}$$

in which, we notice, the first and second columns of the coefficient matrix are interchanged. Given coefficients A_{ij} and the right-hand sides f_i are all numbers, but the list of unknowns tells which coefficient belongs to which unknown. We shall keep the pro forma habit of writing the system as in eq. (1.22), including the list of unknowns, even when they appear in the natural order $x_1, x_2, x_3, \ldots, x_n$.

In an ultimate notational condensation, the entire coefficient matrix is designated by the single letter A, the entire list of unknowns by x, the entire list of right-hand sides by f, and we write

$$Ax = f \tag{1.24}$$

to ideographically symbolize entire system (1.22). Representation (1.24) is of a sufficient descriptive structure to allow for variations of its parts in order to emphasize distinct traits of the system for which it stands. If the right-hand sides of eq. (1.22) are all zeroes, then we write $Ax = o$ to indicate that the system is homogeneous. Similarly, $Ax = f$ and $Ax = f'$, indicate that the two systems of equations have exactly the same coefficients but different right-hand sides, while $Ax = f$ and $A'x = f$ indicate that the two systems have different coefficients but exactly the same right-hand sides.

In the same notational spirit, if f denotes one list of numbers, say the right-hand sides of some systems of linear equations, and f' another, then the concise notation $f = f'$ is shorthand for $f_i = {f_i}'$ for all i. When we say that the two lists f and f' are equal, or the same, we mean that corresponding entries in both lists are equal. Sum $y = x + x'$ of lists x and x' is combined list y such that $y_i = x_i + {x_i}'$ for all i. To see how useful and economical this notational convention is, consider the following theorem: If $Ax = o$ and $Ax' = o$, then $Ay = A(x + x') = o$ as well. Moreover, if $Ax = f$ and $Ax' = f'$, then $f + f' = Ax + Ax'$, giving sense to $Ax + Ax'$.

We shall be careful enough to distinguish contextually between *number* x and *list* x. We shall also find it convenient to abbreviate $(Ax)_i$ for the left-hand side of the ith equation of system $Ax = f$.

Another typographically thrifty notational convention: if $A_{ij} = 0$, its location in the coefficient matrix is left blank,which is also easier on the eye.

1.4 Diagonal and triangular systems

The decoupled, square $n \times n$ system of equations

$$\begin{bmatrix} D_{11} & & & \\ & D_{22} & & \\ & & & \\ & & & D_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \\ f_n \end{bmatrix} \tag{1.25}$$

in which all off-diagonal coefficients D_{ij} $i \neq j$ are zero (blank), is said to be *diagonal*. The solution of the diagonal system is practically read off as

$$x_i = f_i / D_{ii} \qquad i = 1, 2, \ldots, n \tag{1.26}$$

and if none of the diagonal coefficients D_{ii} are zero, then the system possesses a unique solution x.

The (*lower*) *triangular* $n \times n$ system of linear equations

$$\begin{bmatrix} L_{11} & & & & \\ L_{21} & L_{22} & & & \\ L_{31} & L_{32} & L_{33} & & \\ & & & & \\ L_{n1} & L_{n2} & L_{n3} & & L_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \\ x_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \\ f_n \end{bmatrix} \tag{1.27}$$

in which $L_{ij} = 0$ if $j > i$, is also readily solved. The first equation is in only one unknown, and $x_1 = f_1/L_{11}$. Substitution of x_1 into the second equation leaves it with the only unknown x_2, and so on. Hence recursively

$$\begin{aligned} x_1 &= \frac{f_1}{L_{11}} \\ x_2 &= \frac{1}{L_{22}}(f_2 - L_{21}x_1) \\ x_3 &= \frac{1}{L_{33}}(f_3 - L_{31}x_1 - L_{32}x_2) \\ &\vdots \\ x_n &= \frac{1}{L_{nn}}(f_n - L_{n1}x_1 - L_{n2}x_2 - \cdots - L_{n(n-1)}x_{n-1}) \end{aligned} \tag{1.28}$$

and if $L_{ii} \neq 0$ for all i, then formula (1.28) never breaks down and a unique solution is computed for system (1.27).

The underlying idea of the Gauss elimination algorithm for square systems is this: to rework the system of equations, without altering its solutions, into a triangular form, and then if possible into a diagonal form, from which the solution is read off.

Exercises.

1.4.1. Can system

$$\begin{bmatrix} 1 & 3 & -2 \\ & 2 & \\ & -1 & 1 \end{bmatrix} x = f$$

be brought to upper-triangular form by means of equation interchanges only? By means of both equation and unknown interchanges?

1.4.2. Can the lower-triangular system

$$\begin{bmatrix} \times & & & \\ \times & \times & & \\ \times & \times & \times & \\ \times & \times & \times & \times \end{bmatrix} x = f$$

be brought to upper-triangular form by means of equation interchanges only? By means of unknown interchanges only? By means of both?

1.4.3. Can system

$$\begin{bmatrix} & & & \times \\ & & \times & \times \\ & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} x = f$$

be brought to upper-triangular form by means of equation interchanges only? Can it be brought to lower-triangular form by means of unknown interchanges only?

1.4.4. A system that can be brought by means of equation and unknown interchanges to the form

$$\begin{bmatrix} \times & \times & & \\ \times & \times & & \\ & & \times & \times \\ & & \times & \times \end{bmatrix} x = f$$

is said to be *reducible*. It separates in this case into two 2×2, uncoupled systems. Verify that

$$\begin{bmatrix} \times & & \times & \\ & \times & & \times \\ \times & & \times & \\ & \times & & \times \end{bmatrix} x = f$$

is reducible.

1.4.5. Solve

$$\begin{bmatrix} 1 & & & \\ 2 & 2 & & \\ 3 & -1 & 3 & \\ 4 & 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 10 \\ 7 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & -1 & 1 \\ & 1 & -1 & -1 \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 1 \end{bmatrix}.$$

You can verify your answers by substituting the computed values of x back into the original system you started with.

1.5 Equivalence of systems of linear equations -elementary operations

Two systems of linear algebraic equations having the same number of unknowns can share exactly the same solutions even if they have different coefficients, different right-hand sides, and contain a different number of equations. We assume however, that the unknowns are always listed in their natural order.

Definition. *Systems of linear equations are equivalent if they have exactly the same solutions. We write*

$$Ax = f \sim A'x = f' \tag{1.29}$$

if the two systems are equivalent.

Equivalent systems have the same solutions, but some are easier to solve than others. The student may contemplate the fact that the equivalence of systems of linear equations is *transitive*, that is, if $Ax = f \sim A'x = f'$ and $Ax = f \sim A''x = f''$, then $A'x = f' \sim A''x = f''$. We shall replace systems (not necessarily square) of linear equations by equivalent ones with the aid of the three basic *elementary operations*:

1. Interchange of two equations.

2. Multiplication of any equation of the system by a number $\alpha \neq 0$.

3. Addition to any equation of the system, α times any other equation of the system.

We write

$$Ax = f \to A'x = f' \tag{1.30}$$

to indicate that the second system is produced from the first through a finite sequence of elementary operations.

Multiplication of an equation by number α obviously means multiplication of every coefficient of the equation and the right-hand side by α. Addition of two equations obviously means the addition of corresponding coefficients and the addition of the two right-hand sides.

Multiplication of the ith equation by $\alpha_i \neq 0$ (elementary operation number 2), plus a repeated application of elementary operation number 3 results in the more general elementary operation whereby the ith equation of the system is replaced by the *linear combination*

$$\begin{aligned} &\alpha_1 r_1 + \alpha_2 r_2 + \cdots + \alpha_i r_i + \cdots + \alpha_m r_m = 0, \;\; \alpha_i \neq 0 \\ &r_i = (Ax)_i - f_i = A_{i1}x_1 + A_{i2}x_2 + \cdots + A_{in}x_n - f_i \end{aligned} \tag{1.31}$$

of the m equations.

An essential property of elementary operations is expressed in

Theorem 1.1. *Elementary operations are reversible. If an elementary operation transforms $Ax = f$ into $A'x = f'$, then the same kind of elementary operation retrieves $Ax = f$ from $A'x = f'$. More generally, if $Ax = f \rightarrow A'x = f'$, then also $Ax = f \leftarrow A'x = f'$.*

Proof. Notice that had we allowed $\alpha = 0$ in operation 2 the operation would not be reversible, as nothing can be salvaged from $0 = 0$. Otherwise, operation 1 is reversed by the same equation interchange; operation 2 is reversed by multiplying the equation in question by $1/\alpha$; and operation 3 is reversed by adding to the changed equation $-\alpha$ times the equation originally added. If a sequence of elementary operations carries $Ax = f$ into $A'x = f'$, then each operation undone in the reverse order recreates $Ax = f$ out of $A'x = f'$. End of proof.

Theorem 1.2. *The solutions of a given system of linear equations are invariant under elementary operations. In other words, elementary operations produce equivalent systems.*

Proof. Say that elementary operation 1 affected the interchange of the first and the second equations so as to leave us with an original and a transformed system that are now

$$\begin{array}{c} r_1 = 0 \\ r_2 = 0 \\ \vdots \\ r_i = 0 \\ \vdots \\ r_m = 0 \end{array} \qquad \text{and} \qquad \begin{array}{c} r_2 = 0 \\ r_1 = 0 \\ \vdots \\ r_i = A_{i1}x_1 + A_{i2}x_2 + \cdots + A_{in}x_n - f_i = 0 \\ \vdots \\ r_m = 0 \end{array} \quad . \tag{1.32}$$

Since every equation in the original system has a corresponding counterpart in the modified system, and vice versa, every solution of the original system satisfies the modified system and vice versa.

Application of elementary operation 2 leaves us with the original and modified systems

$$\begin{array}{c} r_1 = 0 \\ r_2 = 0 \\ \vdots \\ r_i = 0 \\ \vdots \\ r_m = 0 \end{array} \qquad \text{and} \qquad \begin{array}{c} r_1 = 0 \\ r_2 = 0 \\ \vdots \\ \alpha r_i = 0 \\ \vdots \\ r_m = 0. \end{array} \tag{1.33}$$

Obviously every solution of the original system also satisfies every corresponding equation in the modified system since if $r_i = 0$, then also $\alpha r_i = 0$ for any α. Conversely, if $\alpha r_i = 0$, then, since $\alpha \neq 0$, necessarily $r_i = 0$, and every solution of the modified system also satisfies every corresponding equation in the original system.

Application of elementary operation 3 leaves us with the original and modified systems

$$\begin{array}{c} r_1 = 0 \\ r_2 = 0 \\ \vdots \\ r_i = 0 \\ \vdots \\ r_k = 0 \\ \vdots \\ r_m = 0 \end{array} \quad \text{and} \quad \begin{array}{c} r_1 = 0 \\ r_2 = 0 \\ \vdots \\ r_i = 0 \\ \vdots \\ r_k + \alpha r_i = 0 \\ \vdots \\ r_m = 0 \end{array} \quad . \tag{1.34}$$

Obviously every solution of the original system also satisfies the modified system, whatever α is, since in the original system both $r_i = 0$ and $r_k = 0$. Conversely, any solution of the modified system satisfies all equations of the original system common to both systems, except for possibly the kth equation. But if $r_i = 0$, $i \neq k$, then the kth modified equation reduces to $r_k = 0$, which is also the kth equation in the original system.

Of course, if a system is inconsistent to start with, then it remains so under any sequence of elementary operations. End of proof.

We have shown that elementary operations produce equivalent systems, or that $Ax = f \leftrightarrow A'x = f'$ implies $Ax = f \sim A'x = f'$. We have not shown that two equivalent systems can be obtained from each other through a finite sequence of elementary operations; or that $Ax = f \sim A'x = f'$ implies $Ax = f \leftrightarrow A'x = f'$. This subtler theorem will be proven in due course.

Elementary operations 1,2,3 done on a linear system can be described as manipulations of the entries of the coefficient matrix and list of right-hand sides in tabular form (1.22).

Interchange of, say, equations 1 and 2 of the system is carried out by interchanging rows

1 and 2 of coefficient matrix A, and right-hand sides f_1 and f_2:

$$\begin{bmatrix} A_{21} & A_{22} & & A_{2n} \\ A_{11} & A_{12} & & A_{1n} \\ & & & \\ A_{m1} & A_{m2} & & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \\ x_n \end{bmatrix} = \begin{bmatrix} f_2 \\ f_1 \\ \\ f_m \end{bmatrix}. \tag{1.35}$$

Addition of, say, equation 1 times α_1 to equation 2 is accomplished by the addition of the entries of row 1 of coefficient matrix A times α_1 to the corresponding entries of row 2, and f_1 times α_1 to f_2:

$$\begin{bmatrix} A_{11} & A_{12} & & A_{1n} \\ A_{21}+\alpha_1 A_{11} & A_{22}+\alpha_1 A_{12} & & A_{2n}+\alpha_1 A_{1n} \\ & & & \\ A_{m1} & A_{m2} & & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \\ x_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2+\alpha_1 f_1 \\ \\ f_m \end{bmatrix}. \tag{1.36}$$

Looking at the second row of the coefficient matrix of modified system (1.36) we see that in case $A_{11} \neq 0$ we may choose $\alpha_1 = -A_{21}/A_{11}$ and *eliminate* x_1 from the second equation. The evidence that x_1 is eliminated from equation 2 is a blank first entry in the second row of the modified coefficient matrix.

1.6 Gauss elimination algorithm

A considerable amount of reshuffling, reconstitution and recombination of the entries of the coefficient matrix and right-hand sides takes place during the execution of the Gauss elimination algorithm. To follow the algorithm and draw from it all useful theoretical conclusions, it will be important for us to know which entries are certainly zero, which are possibly zero, and which are certainly nonzero. An actual value of an entry, or the way it got to be there, is procedurally inconsequential.

We have already agreed to mark a zero entry of the coefficient matrix by a blank, and will now add the cross, $\times$, to mark an arbitrary entry that may or may not be zero. An entry that is certainly nonzero is usually assumed to be 1.

Gauss elimination is first described for a square system, which is written in dispatch now as

$$\begin{bmatrix} \underline{\times} & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \\ x_n \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix}. \tag{1.37}$$

We refer to it in

Theorem 1.3. *Every square system of linear equations is equivalent to a triangular system. It can be brought to this form in a finite sequence of elementary operations.*

Proof. A constructive proof is given to this theorem by actually showing how a finite sequence of elementary operations transforms system (1.37) into an equivalent triangular one.

The coefficient of x_1 in the first equation, underlined in the top left corner of the coefficient matrix, or the first diagonal entry, is said to be the first *pivot* of the elimination process. This pivot must be nonzero. If A_{11} happens to be zero, then the first equation is interchanged with the ith equation below it for which it happens that $A_{i1} \neq 0$. At least one equation in the system has a nonzero coefficient of x_1. A first nonzero pivot is thus secured, and α_1 times the first equation is added to the second equation to eliminate x_1 from it. If the coefficient of x_1 in the second equation is zero to start with, then modification of the second equation is skipped and we may go down to the third equation of system (1.37). In this manner, with the aid of the first pivot, x_1 is eliminated from all $n-1$ equations under the first and the resulting equivalent system assumes the form

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ & \underline{\times} & \times & \times & \times \\ & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \\ x_n \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix}. \tag{1.38}$$

As indicated by the list of unknowns, elementary operations make no change in their order of appearance.

Now that the first column is cleared, except for the diagonal coefficient, we turn to the second column of the coefficient matrix and take the second diagonal coefficient to be the next pivot. If it happens to be zero, then the second equation is interchanged with one of the $n-2$ equations *below* it. This equation swap leaves the blanks created in the first column blank. If all coefficients below the second pivot happen to be zero we are done with the column, if not, the nonzero pivot is used to eliminate x_2 from all equations below the 2nd. Again, this elimination does not *fill in* zeroes previously created in the first column. Having cleared the second column under the pivot, we are left with the equivalent system

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & & \underline{\times} & \times & \times \\ & & \times & \times & \times \\ & & \times & \times & \times \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \\ x_n \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix} \tag{1.39}$$

in which the third diagonal entry of the coefficient matrix becomes a pivot candidate. The algorithm cannot fail, and when all the $n-1$ pivots are used up, we invariably end with the upper-triangular system

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & & \times & \times & \times \\ & & & \times & \times \\ & & & & \times \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \\ x_n \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix} \tag{1.40}$$

and the proof is completed.

By choosing the first pivot at the upper left corner of the coefficient matrix, and moving the pivot *down* the diagonal we made the Gauss procedure produce the equivalent *upper*-triangular system of eq. (1.40). To produce a *lower*-triangular system the first pivot is taken at the bottom of the diagonal. Unknown x_n is eliminated by it from all equations *above* the last, then the pivot is moved *up* the diagonal to eliminate the rest of the unknowns.

The first stage of the Gauss algorithm that affects an equivalent upper triangular replacement to the square system is called *forward elimination*, or simply elimination. To continue with the next stage of the procedure, that which reveals the solution, we must distinguish between triangular systems with nonzero diagonal coefficients, and those that have at least one zero diagonal coefficient. For that purpose we make the

Definition. *A triangular (diagonal) system of linear equations is of type 1 if all its diagonal coefficients are different from zero. It is of type 0 if at least one of its diagonal coefficients is zero.*

Observe that the definition of triangular systems of type 0 and 1 rests on inspection rather than on the arithmetic of taking the product of all diagonal coefficients.

We apply another finite sequence of elementary operations to prove

Theorem 1.4. *A triangular system of linear equations of type 1 is equivalent to a diagonal system of the same type.*

Proof. The constructive proof to this algorithm by a repeated application of elementary operations constitutes the second stage of the Gauss elimination algorithm—that of *back substitution*. If a triangular system of linear equations is of type 1, then every equation may be divided by its diagonal coefficient to render it 1, and the system assumes the form

$$\begin{bmatrix} 1 & \times & \times & \times & \times \\ & 1 & \times & \times & \times \\ & & 1 & \times & \times \\ & & & 1 & \times \\ & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \\ x_n \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix}. \tag{1.41}$$

Using coefficient 1 of x_n in the last equation as first pivot, x_n is eliminated from all preceding $n-1$ equations leaving us with

$$\begin{bmatrix} 1 & \times & \times & \times & \\ & 1 & \times & \times & \\ & & 1 & \times & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \\ x_n \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix}. \tag{1.42}$$

The next pivot is coefficient 1 of x_{n-1} in the $(n-1)$th equation of system (1.42), and x_{n-1} is eliminated with this pivot from all $n-2$ preceding equations. Having used up the $n-1$ pivots we are left with the equivalent *diagonal* system of type 1

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \\ x_n \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix} \tag{1.43}$$

and the *unique* solution to the original system is found written in the right-hand side of eq. (1.43).

In order to clearly reveal the sequence of transformations wrought by means of elementary operations leading from the general square system to the triangular, and then if possible to the diagonal, we have painstakingly written the intermediate stages separately. It is understood, however, that in practical computations everything from forward elimination to the end of back substitution can be done *in place*, with a new equivalent system written over the old.

Example. Look carefully at the following sequence of elementary operations intended to quickly and surely bring the original system of linear equations to equivalent upper triangular form:

$$\begin{bmatrix} 0 & 1 & -1 \\ 2 & -1 & 1 \\ 3 & -2 & -4 \end{bmatrix} x = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 1 \\ & 1 & -1 \\ 3 & -2 & -4 \end{bmatrix} x = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 1 \\ & 1 & -1 \\ 6 & -4 & -8 \end{bmatrix} x = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 2 & -1 & 1 \\ & 1 & -1 \\ & -1 & -11 \end{bmatrix} x = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 1 \\ & 1 & -1 \\ & & -12 \end{bmatrix} x = \begin{bmatrix} 3 \\ -1 \\ -6 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 1 \\ & 1 & -1 \\ & & 2 \end{bmatrix} x = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \tag{1.44}$$

and pay heed to the extra elementary operations we undertook to avoid fractional coefficients. At this point we know that the system of equations possesses but one solution and we are prepared to obtain it by back substitutions:

$$\begin{bmatrix} 2 & -1 & 1 \\ & 1 & -1 \\ & & 2 \end{bmatrix} x = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & -2 & 2 \\ & 2 & -2 \\ & & 2 \end{bmatrix} x = \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & -2 & \\ & 2 & \\ & & 2 \end{bmatrix} x = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 4 & & \\ & 2 & \\ & & 2 \end{bmatrix} x = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} \tag{1.45}$$

so that $x_1 = 1, x_2 = -1/2, x_3 = 1/2$, which we verify by substitution into the original system.

Exercises.

1.6.1. Solve both

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ -2 & 1 & 2 \end{bmatrix} x = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & -1 \end{bmatrix} x = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}.$$

1.6.2. Solve

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \end{bmatrix} x = \begin{bmatrix} 2 \\ \\ \\ \\ \\ \end{bmatrix}.$$

1.6.3. Is system

$$\begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix}$$

consistent?

1.6.4. Is system

$$\begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ \\ \\ \end{bmatrix}$$

consistent?

1.6.5. For system

$$\begin{bmatrix} 3 & -1 & 2 \\ 3 & -3 & 4 \\ \alpha & \beta & \gamma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix}$$

what are the conditions on α, β, γ so that its solution is unique? That the system is consistent?

1.7 The echelon and Hermite forms

For a square system of linear equations that is equivalent to a triangular system of type 1, back substitution leaves the ith equation with only one unknown, namely x_i. A zero diagonal at the ith equation in a triangular system of type 0 makes it impossible to eliminate x_i from the equations *above* the ith. Back substitution with *diagonal* pivots is possible only for those diagonal coefficients that are 1, and for a triangular system of type 0 diagonal pivot back substitution is typically exhausted at what is shown below

$$\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} \begin{bmatrix} 1 & \times & & \times & \times & \times & \\ & 0 & & \times & \times & \times & \\ & & 1 & \times & \times & \times & \\ & & & 0 & \times & \times & \\ & & & & 0 & \times & \\ & & & & & 0 & \\ & & & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix}. \tag{1.46}$$

Before continuing the back substitution with *off-diagonal* pivots we make the important observation that *at least one equation in system* (1.46)—*the last equation with a zero diagonal* *coefficient—has all zero coefficients.* If the right-hand side of this equation (equation 6) is not zero then the system, and those derived from it by elementary operations, is inconsistent and insoluble. If the equation turns out to be the correct $0 = 0$, then the system may or may not be consistent. More back substitutions are needed before this is decided.

To continue the back substitution we must resort to off-diagonal pivots. For the sake of clarity we shall interchange equations, taking care not to destroy the upper triangular form of the system, in order to have the pivots always on the diagonal of the coefficient matrix.

In this interchange, equations with a diagonal coefficient equal to 1 stay firm in their places. Only equations with a zero diagonal coefficient may be moved. A fresh supply of, possibly nonzero, pivots is gained by moving down equations 2,4 and 5 of system (1.46) to form the reordered upper-triangular system

$$\begin{matrix} 1 \\ 6 \\ 3 \\ 2 \\ 4 \\ 5 \\ 7 \end{matrix} \begin{bmatrix} 1 & \times & & \times & \times & \times & \\ & 0 & & & & & \\ & & 1 & \times & \times & \times & \\ & & & \times & \times & \times & \\ & & & & \times & \times & \\ & & & & & \times & \\ & & & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix}. \tag{1.47}$$

Equations 1,3 and 7 that have a nonzero diagonal coefficient remain in their place, but equation 6 is moved up to occupy the second position. The sequence of equation orderings that transforms system (1.46) into system (1.47) is 1234567 to 1234657 to 1236457 to 1632457.

Some of the new diagonal entries are zero and some are not. Suppose they are all zero so that system (1.47) is actually

$$\begin{matrix} 1 \\ 6 \\ 3 \\ 2 \\ 4 \\ 5 \\ 7 \end{matrix} \begin{bmatrix} 1 & \times & & \times & \times & \times & \\ & 0 & & & & & \\ & & 1 & \times & \times & \times & \\ & & & 0 & \times & \times & \\ & & & & 0 & \times & \\ & & & & & 0 & \\ & & & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix}. \tag{1.48}$$

Equations are then lowered again (sequence of equation orderings: 1632457 to 1632547 to

1635247) to have diagonal pivot candidates as in the system

$$\begin{array}{c} 1 \\ 6 \\ 3 \\ 5 \\ 2 \\ 4 \\ 7 \end{array} \begin{bmatrix} 1 & \times & & \times & \times & \times & \\ & 0 & & & & & \\ & & 1 & \times & \times & \times & \\ & & & 0 & & & \\ & & & & \times & \times & \\ & & & & & \times & \\ & & & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix}. \tag{1.49}$$

Suppose the diagonal coefficient in equation 2 is zero, but that of equation 4 is nonzero. The newly found nonzero pivot is used now to eliminate x_6 from all equations except the original number 4, and we are left with the equivalent system

$$\begin{array}{c} 1 \\ 6 \\ 3 \\ 5 \\ 2 \\ 4 \\ 7 \end{array} \begin{bmatrix} 1 & \times & & \times & \times & & \\ & 0 & & & & & \\ & & 1 & \times & \times & & \\ & & & 0 & & & \\ & & & & 0 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix}. \tag{1.50}$$

Hermite form

With system (1.50) elimination and back substitution have come to an end since no more pivots are available by equation interchanges that leave the system in upper-triangular form.

Equation (1.50) is the *Hermite* form $Hx = h$ of the system. Hermite form *is a triangular form of type 0 but such that if a diagonal coefficient is zero, then all coefficients in that equation are zero, and if a diagonal coefficient is nonzero, then it is 1 and is the only nonzero coefficient in that column.*

Four zeroes are counted on the diagonal of the coefficient matrix in triangular system (1.46), and one equation (equation 6) is entirely with zero coefficients. But only three zeroes are counted on the diagonal of the coefficient matrix in Hermite system (1.50), and three of its equations (original 2,5,6) have vanishing coefficients. It is clear from the way the Hermite system is created that *if the original triangular system is with m zero diagonal coefficients, then the Hermite form has at least one, but no more than m, zero diagonal coefficients.*

Even though the sequence of elementary operations just carried out to bring upper-triangular system (1.46) into Hermite form (1.50) is specific, it is nevertheless, merely a typical example to a general procedure that transforms any triangular system of type 0 into a Hermite form in a finite number of elementary operations. We formalize this in

Theorem 1.5. *Any triangular system of linear equations of type 0 is practically equivalent to an Hermite system.*

We are now in a position to state

Theorem 1.6. *A necessary and sufficient condition for a Hermite system to have a solution is that the right-hand side is zero whenever a diagonal coefficient is zero.*

Proof. The condition is necessary, for if not, it would entail the contradictory $0 = 1$. It is sufficient, for if we choose $x_i = 0$ whenever the diagonal coefficient of the ith equation is zero, we are left with $x_j = h_j$, $j \neq i$. In other words, if $Hx = h$ is in Hermite form, then $Hh = h$. The right-hand side h of Hermite system $Hx = h$ contains *one* of its solutions. End of proof.

Now that the system is brought into equivalent Hermite form it can be solved. Consider the typical consistent Hermite system,

$$\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \begin{bmatrix} 1 & H_{12} & H_{13} & & H_{15} \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & H_{45} \\ & & & & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} h_1 \\ 0 \\ 0 \\ h_4 \\ 0 \end{bmatrix} \tag{1.51}$$

symbolically written as $Hx = h$. Only equations 1 and 4 need be considered, the other three being the trivial $0 = 0$. Because system (1.51) is in Hermite form, x_1 appears only in equation 1, and x_4 only in equation 4, and they can be written in terms of x_2, x_3, x_5 as

$$\begin{aligned} x_1 &= h_1 - H_{12}x_2 - H_{13}x_3 - H_{15}x_5 \\ x_4 &= h_4 - H_{45}x_5 \end{aligned} \tag{1.52}$$

or

$$\begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_4 \end{bmatrix} + \begin{bmatrix} -H_{12} & -H_{13} & -H_{15} \\ & & -H_{45} \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_5 \end{bmatrix}. \tag{1.53}$$

We may select any values for x_2, x_3, x_5 and get corresponding values for x_1, x_4, and any such five numbers constitute a solution of system (1.51) and any other system equivalent to it. The three unknowns x_2, x_3, x_5 that have a zero diagonal coefficient are *free, arbitrary*, or *independent*, while unknowns x_1, x_4 that have a one for their diagonal coefficient are *dependent*. If, say, $H_{12} \neq 0$, then x_1, x_3, x_5 could have been chosen as independent unknowns, and x_2, x_4 as dependent.

Theorem 1.7. *A system of linear equations that is equivalent to a triangular system of type* 1 *possess a unique solution. A system of linear equations that is equivalent to a triangular system of type* 0 *possess more than one solution*

Proof. A system that is equivalent to an upper triangular system of type 1 is ultimately equivalent to the Hermite system $x_i = h_i$ for all i from 1 to n. A system that is equivalent to an upper triangular system of type 0 is ultimately equivalent to a Hermite form of the same type with at least one $0 = 0$ equation. At least one unknown of this system is arbitrary and can be assigned any numerical value at will. End of proof.

Examples.

1. System

$$\begin{bmatrix} 1 & & -1 & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1.54}$$

is in Hermite form and is inclusively solved by $x_1 = 1 + x_3,\ x_2 = x_2,\ x_3 = x_3,\ x_4 = 0$. Unknowns x_2 and x_3 of the system are arbitrary or independent, while unknowns x_1 and x_4 of the system are dependent.

2. The following elementary operations carry an upper-triangular matrix of type 0 into its equivalent Hermite form:

$$\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 1 \\ & 0 & 1 & 2 \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} x = \begin{bmatrix} \\ -1 \\ \\ \\ \end{bmatrix} \rightarrow \begin{matrix} 1 \\ 4 \\ 2 \\ 3 \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 1 \\ & 0 & & \\ & & 1 & 2 \\ & & & 1 \end{bmatrix} x = \begin{bmatrix} \\ \\ -1 \\ \\ \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 2 & -1 & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} x = \begin{bmatrix} \\ \\ -1 \\ \\ \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} x = \begin{bmatrix} -1 \\ \\ -1 \\ \\ \end{bmatrix} \tag{1.55}$$

so that $x_1 = -1 - 2x_2,\ x_2 = x_2,\ x_3 = -1,\ x_4 = 0$.

Removing the redundant $0 = 0$ equations from the consistent Hermite form (1.50) and compressing the system transforms it to the *reduced echelon* form below

$$\begin{matrix} 1 \\ 3 \\ 4 \\ 7 \end{matrix} \begin{bmatrix} 1 & \times & & \times & \times & & \\ & & 1 & \times & \times & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \end{bmatrix}. \tag{1.56}$$

Reduced echelon form

Notice that in this form the equations are arranged in ascending order of zero coefficients before the leading 1 coefficient.

The nonreduced echelon form corresponding to eq. (1.56) is

$$\begin{matrix} 1 \\ 3 \\ 4 \\ 7 \end{matrix} \begin{bmatrix} 1 & \times & \times & \times & \times & \times & \times \\ & & 1 & \times & \times & \times & \times \\ & & & & & 1 & \times \\ & & & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \end{bmatrix}. \tag{1.57}$$

In this form, as in form (1.56), the equations of the system are ordered in a *strictly increasing* order of the number of zero coefficients before the first nonzero coefficient in each equation. But the leading 1 coefficient is not the only nonzero coefficient in its matrix column.

Hermite form is specifically square, but evidently any rectangular system can be brought to echelon form by a finite sequence of elementary operations.

Being at the end of a chain of elementary operations, the Hermite form, and hence the reduced echelon form, of a consistent system of linear equations has special properties.

Lemma 1.8. *If all unknowns in an $m \times n$ system of equations are arbitrary, then the system must be a null system consisting of only 0=0 equations.*

Proof. Let the system be

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 &= f_1 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 &= f_2. \end{aligned} \tag{1.58}$$

Since x_1, x_2, x_3 are arbitrary we may assign them any numerical value. Choosing $x_1 = x_2 = x_3 = 0$ we find that $f_1 = f_2 = 0$. Choosing next $x_1 = 1, x_2 = x_3 = 0$ we verify that $A_{11} = A_{21} = 0$. With $x_2 = 1, x_3 = 0$ and then $x_3 = 1$, we finally establish that all coefficients and right-hand sides of the system are zero. End of proof.

Theorem 1.9. *Equivalent systems have the same Hermite form.*

Proof. Let $Hx = h$ and $H'x = h'$ be two consistent Hermite systems such that $Ax = f \sim Hx = h$, and $A'x = f' \sim H'x = h'$, where $Ax = f \sim A'x = f'$. Then $Hx = h \sim H'x = h'$, and we shall show that $H_{ij} = H_{ij}'$ and $h_i = h_i{}'$.

Still with reference to the typical $Hx = h$ in eq. (1.51) suppose that Hermite system $H'x = h'$

$$\begin{matrix}1\\2\\3\\4\\5\end{matrix}\begin{bmatrix} H_{11}' & H_{12}' & H_{13}' & H_{14}' & H_{15}' \\ & H_{22}' & H_{23}' & H_{24}' & H_{25}' \\ & & H_{33}' & H_{34}' & H_{35}' \\ & & & H_{44}' & H_{45}' \\ & & & & H_{55}' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} h_1' \\ h_2' \\ h_3' \\ h_4' \\ h_5' \end{bmatrix} \tag{1.59}$$

is equivalent to it. We shall show that if systems (1.51) and (1.59) have exactly the same solutions, then they are one and the same.

The arbitrary choice $x_2 = x_3 = x_5 = 0$ reduces eq. (1.59) to

$$\begin{bmatrix} H_{11}' & H_{14}' \\ & H_{44}' \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} h_1' \\ h_4' \end{bmatrix} \tag{1.60}$$

which must be uniquely solved by $x_1 = h_1,\ x_2 = h_2$. Hence it must happen that $H_{11}' = H_{44}' = 1$ and it follows further from the assumption on the Hermite form of system (1.59) that $H_{14}' = H_{24}' = H_{34}' = 0$, and that $h_1' = h_1,\ h_2' = h_2$.

Substitution of x_1 and x_4 from eq. (1.52) into equations 2,3,5 of system (1.59) produces

$$\begin{aligned} H_{22}'x_2 + H_{23}'x_3 + H_{25}'x_5 &= h_2' \\ H_{33}'x_3 + H_{35}'x_5 &= h_3' \\ H_{55}'x_5 &= h_5' \end{aligned} \tag{1.61}$$

which holds for *arbitrary* x_2, x_3, x_5. By lemma 1.8 this can happen only if the coefficients and right-hand sides of eq. (1.61) are all zero, and it follows that equations 2,3,5 of $H'x = h'$ are 0=0, the same as in $Hx = h$.

Finally, substitution of x_1 and x_4 from eq. (1.52) into equations 1 and 4 of $H'x = h'$ yields

$$\begin{aligned}(H_{12}{}' - H_{12})x_2 + (H_{13}{}' - H_{13})x_3 + (H_{15}{}' - H_{15})x_5 = 0 \\ (H_{45}{}' - H_{45})x_5 = 0\end{aligned} \tag{1.62}$$

and, again, for the system to hold true for arbitrary x_2, x_3, x_5 all its coefficients must vanish. Consequently $H_{12}{}' = H_{12},\ H_{13}{}' = H_{13},\ H_{15}{}' = H_{15},\ H_{45}{}' = H_{45}$, and Hermite form (1.51) is recovered. End of proof.

Theorem 1.9 can be formulated in terms of elementary operations instead of equivalence.

Theorem 1.10. *The Hermite, and hence also the reduced echelon, form of a consistent system is uniquely determined in a finite sequence of elementary operations.*

Proof. We have demonstrated in the beginning of this section that any square system can be brought to Hermite form in a finite number of elementary operations. Deletion of the 0=0 equations in a consistent Hermite system reduces it to echelon form. Suppose that finite sequences of elementary operations produce the two Hermite systems $Hx = h$ and $H'x = h'$ out of the same consistent system $Ax = f$. Symbolically

$$Ax = f \begin{array}{l} \nearrow\!\!\!\swarrow \ Hx = h \\ \nwarrow\!\!\!\searrow \ H'x = h' \end{array} . \tag{1.63}$$

Then there are sequences of elementary operations that turn $Hx = h$ into $H'x = h'$ and vice versa; $Hx = h \leftrightarrow H'x = h'$. They are

$$Ax = f \begin{array}{l} \nearrow \ Hx = h \\ \nwarrow \ H'x = h' \end{array} \quad \text{and} \quad Ax = f \begin{array}{l} \swarrow \ Hx = h \\ \searrow \ H'x = h' \end{array} . \tag{1.64}$$

We shall show that the most general elementary operation that replaces $Hx = h$ by $H'x = h'$ actually leaves the system unaltered.

None of the equations of $Hx = h$, we observe, can be multiplied by a number different from 1, except for the trivials $0 = 0$, if the system is to remain in Hermite form. Interchange of equations is also permissible only in the trivial case of two $0 = 0$ equations. Look at Hermite system (1.51) for a cursory verification of this claim.

Otherwise, the most general elementary operation consists of replacing an equation of the system by the linear combination (1.31) of all equations in the system. Starting with the last equation of $Hx = h$ in eq. (1.51) and working our way up we replace each of the equations by a linear combination of all equations, while making sure that the replaced system stays in Hermite form.

Equation 5 is replaced by the linear combination of α_1 times equation 1, plus α_2 times equation 2 and so on. Because equations 2,3,5 of $Hx = h$ are $0 = 0$, the new equation 5 is

$$\begin{array}{c}\alpha_1 x_1 + \alpha_1 H_{12} x_2 + \alpha_1 H_{13} x_3 + \alpha_4 x_4 + (\alpha_1 H_{15} + \alpha_4 H_{45}) x_5 \\ = \alpha_1 h_1 + \alpha_4 h_4\end{array}. \tag{1.65}$$

For the system to remain upper-triangular it is necessary that $\alpha_1 = 0$ (because $H_{11} = 1$), and $\alpha_4 = 0$ (because $H_{44} = 1$), and the 5th equation reverts to the old $0 = 0$. Elementary operations are sequential; they operate on the modified system. But since equation 5 is unchanged, equation 4 is replaced by the same linear combination (1.65). Upper-triangular form dictates also for the 4th modified equation $\alpha_1 = 0$, and it becomes

$$\alpha_4 x_4 + \alpha_4 H_{45} x_5 = \alpha_4 h_4. \tag{1.66}$$

No equation interchange is possible, and in order to have a legitimate elementary operation we are forced to set $\alpha_4 = 1$ in eq. (1.66), recovering the original equation 4 of $Hx = h$. We have come up to the third equation, which we replace by the linear combination (1.65). Upper-triangular form dictates that α_1 be zero, and $\alpha_4 = 0$ because the coefficients of x_4 in all equations except the 4th need be zero for Hermite form. The 3rd modified equation turns out to be $0 = 0$. Continuing in the same manner we conclude that Hermite form $Hx = h$ remains unchanged.

The consistency assumption of this theorem is necessary, for if one of the "equations" in the Hermite system is $0 = 1$, then multiplying it by $\alpha \neq 0$ produces $0 = \alpha$ which leaves the system in a Hermite form, but with a different right-hand side. End of proof.

Theorem 1.11. *If two consistent systems of linear equations are equivalent, then one is obtainable from the other by a finite sequence of elementary operations.*

Proof. Let $Ax = f \sim A'x = f'$ be the two systems. According to theorems 1.1, 1.9 and 1.10

$$\begin{array}{rcl} Ax = f & \nwarrow\!\!\searrow & \\ & & Hx = h \\ A'x = f' & \swarrow\!\!\nearrow & \end{array} \tag{1.67}$$

where $Hx = h$ is the Hermite system common to both. Then

$$\begin{array}{rcl} Ax = f & \searrow & \\ & & Hx = h \\ A'x = f' & \swarrow & \end{array} \quad \text{and} \quad \begin{array}{rcl} Ax = f & \nwarrow & \\ & & Hx = h \\ A'x = f' & \nearrow & \end{array} \tag{1.68}$$

imply $Ax = f \leftrightarrow A'x = f'$. End of proof.

Uniqueness entitles the Hermite and reduced echelon forms to the appellation *canonical*.

Definition. *The rank of a consistent system is the number of nonzero diagonal coefficients in its Hermite form, or the number of equations, excluding* $0 = 0$*, in its echelon form. A consistent linear system that has no* $0 = 0$ *redundancies is said to be of full rank.*

A square system of equations is of full rank if and only if it is equivalent to an upper-triangular system of type 1.

Rank, which also counts the number of dependent unknowns in the system, is theoretically valuable and terminologically indicative. Equivalent systems have the same rank. If system $Ax = f$ is of rank r, then there exists a system of equations $A'x = f'$ equivalent to it with r equations, not fewer. If system $Ax = f$ with m equations is of rank r, then some $m - r$ equations may be deleted from the system without altering the solution, not more.

Exercises.

1.7.1. Bring to Hermite form, then to reduced echelon form

$$\begin{bmatrix} 1 & -2 & & 2 & \\ & & & & 1 \\ & & 1 & -3 & \end{bmatrix} x = o.$$

1.7.2. Write the solution of

$$\begin{bmatrix} 1 & 1 & -1 & & -1 & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 1 & 1 & \\ & & & & 0 & \\ & & & & & 1 \end{bmatrix} x = o.$$

1.7.3. What is the rank of

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 4 \end{bmatrix} x = o\ ?$$

1.7.4. What is the rank of

$$\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} x = o\ ?$$

1.7.5. What are the ranks of

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} x = o \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 1 & 1 \\ & 0 & 1 & 1 \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} x = o\ ?$$

1.7.6. Is it true that the rank of

$$\begin{bmatrix} 1 & \times & \times \\ & 1 & \times \\ & & 0 \end{bmatrix} x = o \text{ is higher than the rank of } \begin{bmatrix} 1 & \times & \times \\ & 0 & \times \\ & & 0 \end{bmatrix} x = o\ ?$$

1.7.7. If *one* coefficient in linear system $Ax = o$ is changed, how much lower can the rank of the changed system be?

1.7.8. Let $a_1 \neq 0,\ a_2 \neq 0,\ a_3 \neq 0$, and show that the rank of

$$\begin{bmatrix} a_1a_1 & a_1a_2 & a_1a_3 \\ a_2a_1 & a_2a_2 & a_2a_3 \\ a_3a_1 & a_3a_2 & a_3a_3 \end{bmatrix} x = o$$

is 1.

1.7.9. Bring system

$$\begin{bmatrix} 1 & -2 & 3 & 6 & -8 \\ & 0 & 0 & 7 & -5 \\ & & 0 & 13 & -4 \\ & & & 9 & 4 \\ & & & & -3 \end{bmatrix} x = o$$

to Hermite form. Show that its solution is $x_1 = 2\alpha - 3\beta$, $x_2 = \alpha,\ x_3 = \beta,\ x_4 = 0,\ x_5 = 0$.

1.7.10. Bring to Hermite form and solve

$$\begin{bmatrix} 1 & -1 & 2 & 1 & -7 \\ & 0 & 0 & -1 & 5 \\ & & 0 & 2 & 9 \\ & & & 0 & 3 \\ & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \\ -5 \\ -3 \\ -1 \end{bmatrix}.$$

Particular solution: $x_1 = 1, x_2 = 1,\ x_3 = -1, x_4 = 2, x_5 = -1$.

1.7.11. Bring system

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 1 & 2 & 1 & 2 \\ 3 & 6 & 1 & 0 \\ 2 & 4 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 7 \\ -7 \end{bmatrix}$$

Particular solution: $x_1 = -1,\ x_2 = 1,\ x_3 = 4,\ x_4 = 1$.

1.7.12. Bring to Hermite form and solve

$$\begin{bmatrix} 1 & -1 & 3 & 1 & 2 \\ 3 & 0 & 4 & -1 & 3 \\ 2 & 1 & 1 & -2 & 1 \\ 5 & 1 & 5 & -2 & 3 \\ 3 & 0 & 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 3 \\ 12 \\ 9 \end{bmatrix}.$$

Particular solution: $x_1 = x_2 = x_3 = x_4 = x_5 = 1$.

1.7.13. In the Hermite system

$$\begin{bmatrix} 1 & & -3 & \\ & 1 & & \\ & & 0 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ 0 \\ h_4 \end{bmatrix}$$

interchange x_1 with x_4, then bring the transformed system back to Hermite form.

1.7.14. Are systems

$$\begin{bmatrix} 1 & -2 & 3 \\ -3 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & -5 \\ -1 & -3 & 7 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

equivalent? If yes, find the sequence of elementary operations that transforms the first into the second.

1.7.15. Are systems

$$\begin{bmatrix}1 & -1 & 2\\ & 1 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \text{ and } \begin{bmatrix}2 & -2 & 4\\ 1 & 1 & 8\\ -2 & 5 & 5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

equivalent?

1.8 Homogeneous square systems

A homogeneous system of linear equations is symbolized by $Ax = o$ to indicate that the right-hand sides are without exception zero. Every homogeneous system, whether square or not, is consistent and admits the *trivial solution* $x_1 = x_2 = \cdots = x_n = 0$. Under what conditions does a square homogeneous system of linear equations possess a *nontrivial* solution, a solution in which at least one of the unknowns is not zero? For an answer we have

Theorem 1.12. *A necessary and sufficient condition that a square homogeneous system of linear equations have a nontrivial solution is that it be equivalent to a triangular system of type* 0.

Proof. The condition is necessary since a square system that is equivalent to a triangular system of type 1 has a unique solution, here the trivial $x_1 = x_2 = \cdots = x_n = 0$. The condition is also sufficient since according to theorem 1.7 such a system possesses multiple solutions with arbitrary unknowns that may be chosen different from zero. End of proof.

Theorem 1.13. *If the trivial solution is the only solution of the square homogeneous system $Ax = o$, then the nonhomogeneous $Ax = f$ is soluble, and the solution is unique.*

Proof. The interest of the theorem is in its wording. It states that *uniqueness* in the homogeneous $Ax = o$ implies *existence* for the nonhomogeneous $Ax = f$; and these are two of the most exalted words in all mathematics.

If $Ax = o$ has a unique solution, then the system is equivalent to a triangular form of type 1, and this in turn is necessary and sufficient for $Ax = f$ to have a unique solution. End of proof.

Exercises.

1.8.1. Bring system

$$\begin{bmatrix} 1 & 1 & \alpha \\ -3 & 5 & \alpha \\ \alpha & -1 & 3 \end{bmatrix} x = o$$

to equivalent upper-triangular form. For what values of α does the system have nontrivial (multiple) solutions?

1.9 Rectangular systems -more unknowns than equations

Any rectangular system with more unknowns than equations can obviously be brought to a reduced echelon form. In fact, a rectangular system with more unknowns than equations can be looked upon as square, with the addition of the needed number of null, $0 = 0$, equations.

The next is a principal theorem on systems of linear equations.

Theorem 1.14. *A homogeneous system of linear equations with more unknowns than equations has a nontrivial solution.*

Proof. With the addition of the proper number of $0 = 0$ equations the rectangular system is made square. The augmented system is certainly equivalent to a homogeneous triangular system of type 0. According to theorem 1.7 such a system possesses multiple solutions with arbitrary unknowns that may be chosen different from zero. End of proof.

Definition. *If nonhomogeneous system $Ax = f$ is consistent, then any one of its solutions is said to be a particular solution.*

Theorem 1.15. *If x' is a particular solution of $Ax = f$, then any other solution to this system is $x' + x''$, where x'' is a solution of $Ax = o$.*

Proof. Let x' be a particular solution and split x into $x'+x''$, so as to have $Ax'+Ax''=f$. Since $Ax'=f$ we are left with $Ax''=o$. End of proof.

The separation of the solutions to the $m \times n$ system $Ax=f$ into the sum of a homogeneous part and a nonhomogeneous part is an immediate consequence of the linearity of the system. Theorem 1.15 is simple to the point of being almost self-evident, yet it is useful. If a particular solution is available to the nonhomogeneous system $Ax=f$, then the solution to this system is completed with the addition of all solutions of the homogeneous system $Ax=o$.

Example.

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -3 \\ 3 & 4 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}, x' = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -3 \\ 3 & 4 & 1 \end{bmatrix} x^{''} = 0 \tag{1.69}$$

$$\begin{matrix} x_1 = 4-3\alpha \\ x_2 = -1+2\alpha \\ x_3 = \alpha \end{matrix} \;,\; \begin{matrix} x_1^{''} = -3\beta \\ x_2^{''} = 2\beta \\ x_3^{''} = \beta \end{matrix} \;,\; \begin{matrix} x_1 = 1-3\beta \\ x_2 = 1+2\beta \\ x_3 = 1+\beta \end{matrix} \;,\; \alpha = 1+\beta. \tag{1.70}$$

The previous theorem also has an interesting corollary.

Corollary 1.16. *Let $Ax=f$ be an $m \times n$ consistent nonhomogeneous system of linear equations. If $Ax=o$ has a unique solution, then so does $Ax=f$, and if $Ax=o$ has more than one solution, then so does $Ax=f$. In other words, a consistent system of equations with more unknowns than equations has multiple solutions.*

Proof. Since $Ax=f$ is consistent it has at least one particular solution x'. Let $x^{''}$ be any solution of $Ax=o$. If $x^{''}=o$ is the unique solution to the homogeneous system, then by the previous theorem $x=x'+o$ is the only solution to $Ax=f$. On the other hand, if $x^{''}$ is not unique, then the solution $x=x'+x^{''}$ to $Ax=f$ is not unique. End of proof.

Corollary 1.16 could also be proved with reference to the Hermite form of the system completed to square by the addition of the right number of $0=0$ equations.

Rectangular systems are solved, just as square systems are, by first bringing the system to the reduced echelon form

$$\begin{bmatrix} 1 & \times & & \times & & \times & & & & \times \\ & & 1 & \times & & \times & & & & \times \\ & & & & 1 & \times & & & & \times \\ & & & & & & 1 & & & \times \\ & & & & & & & 1 & & \times \\ & & & & & & & & 1 & \times \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix} \tag{1.71}$$

and then separating the unknowns into dependent and independent. Unknowns for which the leading 1 in each equation is their coefficient are taken as dependent, the rest are independent. In eq. (1.71) $x_1, x_3, x_5, x_7, x_8, x_9$ are dependent, and x_2, x_4, x_6, x_{10} are independent. A system of rank r has r dependent and $n - r$ independent unknowns. Moving the terms of eq. (1.71) with the independent unknowns to the right results in

$$\begin{bmatrix} x_1 \\ x_3 \\ x_5 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} \times & \times & \times & \times \\ & \times & \times & \times \\ & & \times & \times \\ & & & \times \\ & & & \times \\ & & & \times \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \\ x_6 \\ x_{10} \end{bmatrix} + \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix} \tag{1.72}$$

which is the ultimate goal of the solution procedure.

Or we may set $x_2 = z_1, x_4 = z_2, x_6 = z_3, x_{10} = z_4$ and write the solution as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix} = \begin{bmatrix} \times & \times & \times & \times \\ 1 & & & \\ & \times & \times & \times \\ & 1 & & \\ & & \times & \times \\ & & 1 & \\ & & & \times \\ & & & \times \\ & & & \times \\ & & & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} \times \\ \\ \times \\ \\ \times \\ \\ \times \\ \times \\ \times \\ \\ \end{bmatrix} \tag{1.73}$$

expressing all x's in terms of the right-hand sides and the four *variables* z_1, z_2, z_3, z_4.

A consistent rectangular system with more unknowns than equations has always at least one arbitrary unknown, and hence has more than one solution. Linear systems with more than one solution are said to be *underdetermined*.

Exercises.

1.9.1. Argue that if x solves the homogeneous system $Ax = o$, then αx solves it too.

1.9.2. Solve

$$\begin{bmatrix} 1 & 2 & 1 & -3 \\ -1 & -2 & 3 & 3 \\ 3 & 6 & -5 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

1.9.3. Solve

$$\begin{bmatrix} 1 & 2 & -1 & 3 & 1 \\ 1 & 4 & -3 & 5 & -1 \\ 1 & 3 & -2 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}.$$

1.9.4. Nonhomogeneous system

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ -1 & 6 & -7 & -1 \\ -1 & 2 & -2 & -1 \\ 1 & 6 & -8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -2 \\ 0 \end{bmatrix}$$

is seen to possess the particular solution $x_1{}' = x_2{}' = x_3{}' = x_4{}' = 1$. Solve $Ax^{''} = o$ and show that $x = x' + x^{''}$.

1.10 Rectangular systems -more equations than unknowns

Forward elimination invariably brings the rectangular system to the equivalent form

$$\begin{array}{cc} \uparrow & \uparrow \\ & n \\ m & \downarrow \\ & \uparrow \\ & m-n \\ \downarrow & \downarrow \end{array} \overset{\leftarrow \quad n \quad \rightarrow}{\begin{bmatrix} \times & \times & \times \\ & \times & \times \\ & & \times \\ & & \\ & & \\ & & \end{bmatrix}} \begin{bmatrix} x_1 \\ \\ x_n \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix}. \tag{1.74}$$

A necessary, though of course not sufficient, condition for the existence of a solution for system (1.74) is that the last $m - n$ equations are $0 = 0$— that the system reduces to a triangular form. A system of equations in which the inconsistency $0 = 1$ is discovered is occasionally said to be *overdetermined*.

A consistent $m \times n,\ m > n$, system of equations certainly includes $m - n$ redundant, $0 = 0$, equations, and certainly cannot be of full rank.

Exercises.

1.10.1. Solve

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

1.10.2. Solve

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & -2 & 3 \\ 4 & 1 & 4 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 9 \\ 3 \end{bmatrix}.$$

Particular solution: $x_1 = 1,\ x_2 = 1,\ x_3 = 1$.

1.11 Summary

With this, our procedural discussion of the solution of the linear algebraic system has come to an end, and we summarize it as follows:

1. *Square system* $Ax = f$ is first brought by means of elementary operations to upper-triangular form, and the diagonal coefficients are inspected. A triangular system of type 1 is consistent and possesses a unique solution, obtained by back substitution. A triangular system of type 0 may or may not be consistent, but if consistent, then it possesses multiple solutions. Homogeneous $Ax = o$ is consistent, and its equivalence to an upper-triangular system of type 0 implies the existence of nontrivial solutions.

Further elementary operations transform an upper-triangular form of type zero into Hermite form, and if no $0 = 1$ absurdities are found, then the system is consistent, and allows writing the dependent unknowns in terms of the independent. The number of nonzero diagonal coefficients in the Hermite form, the rank of the system, equals the number of dependent unknowns.

2. *Rectangular system* $Ax = f$, with more unknowns than equations, may be considered square with the addition of the right number of 0=0 equations. Such a system is certainly

equivalent to an upper-triangular system of type 0, and if the system is consistent, then it possesses many solutions. A system of one equation in n unknowns is consistent, as a contradiction cannot arise between fewer than two equations. Homogeneous system $Ax = o$ is always consistent, and possesses a nontrivial solution if the number of unknowns exceeds the number of equations. A system of linear equations harboring no 0=0 redundancies is said to be of full rank.

3. *Rectangular system* $Ax = f$ of m equations in n unknowns, $m > n$, if consistent, includes at least $m - n, 0 = 0$, redundancies. The rank of an $m \times n$ system, $m > n$, cannot exceed n. The first $m - n$ redundancies are discovered in the forward elimination phase of the solution. Their deletion leaves the system in upper-triangular form and we proceed with it as with an originally square system.

1.12 Arbitrary right-hand sides

Solution of square system $Ax = f$, having a unique answer, is divided in this chapter into the two major steps, both done with elementary operations, of forward elimination and back substitution. In the first step the system is brought to triangular form, and in the second to diagonal. Elementary operations were done until now under the assumption that both the coefficient matrix and the right-hand sides are given numerically.

In this section we consider forward elimination and back substitution for systems with arbitrary, or variable, right-hand sides. But first we prove

Theorem 1.17. *Equation orderings exist for linear system $Ax = f$, with which forward elimination is accomplished without interchanges.*

Proof. If in linear system $Ax = f$ $A_{11} = 0$, then the first equation is interchanged with one of the equations below it for which $A_{i1} \neq 0$. Say then that $A_{11} \neq 0$ and use it to eliminate x_1 so as to have

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} + \alpha_1 A_{12} & A_{23} + \alpha_1 A_{13} & A_{24} + \alpha_1 A_{14} \\ 0 & A_{32} + \alpha_2 A_{12} & A_{33} + \alpha_2 A_{13} & A_{34} + \alpha_2 A_{14} \\ 0 & A_{42} + \alpha_3 A_{12} & A_{43} + \alpha_3 A_{13} & A_{44} + \alpha_3 A_{14} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 + \alpha_1 f_1 \\ f_3 + \alpha_2 f_1 \\ f_4 + \alpha_3 f_1 \end{bmatrix}. \tag{1.75}$$

Where $\alpha_1 = -(A_{21}/A_{11}), \alpha_2 = -(A_{31}/A_{11}), \alpha_3 = -(A_{41}/A_{11})$. An equation swap among the equations *below* the first might again be needed to have a nonzero second pivot. But this equation interchange could have been done before x_1 is eliminated, for once the first equation is decided, the coefficients of x_2 before and after elimination of x_1 are fixed, and all the equation interchange does is move them up and down the second matrix column. Once the first and second equations are chosen the coefficients of x_3 become decided, and any equation interchanges, excluding equations 1 and 2, can be done before elimination. Every subsequent equation interchange does not involve the equations already ordered, and could have been done before the start of elimination. Hence, it is possible to order the equations of the system initially so that forward elimination proceeds from start to finish without intermediate equation interchanges. End of proof.

We shall assume that linear system $Ax = f$ is ordered to avoid equation interchanges in forward elimination. The student should have no difficulties now in accepting the writing of

$$\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \\ f_n \end{bmatrix} \tag{1.76}$$

for the linear system $Ax = f$ with variable right-hand sides. In a sense, equation (1.76) is written symmetrically for x and f and multiplication of an equation by a number is performed by multiplying the coefficients of x and f on both sides, while addition of two equations means the addition of corresponding coefficients on both sides.

Forward elimination without equation interchanges produces

$$\begin{bmatrix} \times & \times & \times & \times \\ & \times & \times & \times \\ & & \times & \times \\ & & & \times \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} = \begin{bmatrix} 1 & & & \\ \times & 1 & & \\ \times & \times & 1 & \\ \times & \times & \times & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \\ f_n \end{bmatrix} \tag{1.77}$$

or in short $Ux = Lf$. If the upper-triangular system is of type 1, then back substitution produces

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \\ f_n \end{bmatrix} \tag{1.78}$$

or in short $x = Bf$.

In equation $f = Ax$, f is linearly expressed in terms of x, while in equation $x = Bf$, x is linearly expressed in terms of f, and the original system is *inverted*.

To write the general solution of *rectangular* system $Ax = f$ we assume that it is of full rank with all $0 = 0$ removed. We start with

$$\begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{bmatrix} x = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} f \tag{1.79}$$

and bring the system by means of elementary operations into the form

$$\begin{bmatrix} 1 & & & & A_{15}' & A_{16}' \\ & 1 & & & A_{25}' & A_{26}' \\ & & 1 & & A_{35}' & A_{36}' \\ & & & 1 & A_{45}' & A_{46}' \end{bmatrix} x = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix} f. \tag{1.80}$$

Admittedly, it does not always happen that the independent unknowns are grouped last, but this does not detract from the generality of the discussion since the unknowns may be reordered to appear this way. Unknowns x_1, x_2, x_3, x_4 are readily written now in terms of f_1, f_2, f_3, f_4 and arbitrary x_5 and x_6. We prefer, however, to write $x_5 = z_1$, $x_6 = z_2$ and write the solution as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} - \begin{bmatrix} A_{15}' & A_{16}' \\ A_{25}' & A_{26}' \\ A_{35}' & A_{36}' \\ A_{45}' & A_{46}' \\ -1 & \\ & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{1.81}$$

in which, as said, z_1 and z_2 are variables that can assume independently any real value.

Exercises.

1.12.1. Write

$$\begin{bmatrix} 1 & & \\ -1 & 2 & \\ 2 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

and invert the system to have $x = Bf$. Hint: If $x_1 = x_2 = x_3 = 1$, then $f_1 = 1, f_2 = 1, f_3 = -1$.

1.12.2. Write

$$\begin{bmatrix} -2 & 1 & -1 \\ & 2 & -2 \\ & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

and invert the system to have $x = Bf$. Hint: If $x_1 = x_2 = x_3 = 1$, then $f_1 = -2, f_2 = 0, f_3 = 1$.

1.12.3. Write

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 0 \\ -2 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

and invert the system to have $x = Bf$. Hint: If $x_1 = 1,\ x_2 = -1,\ x_3 = 1$, then $f_1 = -2,\ f_2 = 2,\ f_3 = -6$.

1.12.4. Invert the system

$$\begin{bmatrix} & 1 & & \\ 1 & & 1 & \\ & 1 & & 1 \\ & & 1 & \end{bmatrix} x = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} f.$$

1.12.5. For what values of α does system

$$\begin{bmatrix} 1 & 1 & \alpha \\ 1 & \alpha & 1 \\ \alpha & 1 & 1 \end{bmatrix} x = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} f$$

have a unique solution for any f?

1.12.6. For what f_1, f_2, f_3 is system

$$\begin{bmatrix} 1 & -1 & \\ -1 & 2 & -1 \\ & -1 & 1 \end{bmatrix} x = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

consistent?

1.12.7. For what values of f_1, f_2, f_3, f_4 is system

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 3 & 2 & 3 & 2 \\ 5 & 5 & 4 & 1 \\ 2 & 3 & 1 & -1 \end{bmatrix} x = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

consistent?

1.12.8. Write

$$\begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix}$$

and invert the system to have $x = Bf$.

1.12.9. For the system

$$x_1 + 2x_2 = f_1$$

$$-2x_1 + x_2 = f_2$$

with arbitrary f_1, f_2, assume that

$$x_1 = B_{11}f_1 + B_{12}f_2$$

$$x_2 = B_{21}f_1 + B_{22}f_2.$$

Then substitute x_1 and x_2 from this system into the first system and compute B_{11}, B_{12}, B_{21}, B_{22} by Lemma 1.8.

1.12.10. For system

$$x_1 - x_2 = f_1$$

$$-x_1 + x_2 = f_2$$

write

$$x_1 = B_{11}f_1 + B_{12}f_2$$

$$x_2 = B_{21}f_1 + B_{22}f_2$$

and show by substitution that no such general solution exists.

1.12.11. System

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

possesses the solutions $x_1 = x_2 = 1$, and $x_1 = -1, x_2 = 3$. What are its coefficients?

1.12.12. The (2×2) system $Ax = f$ is such that when $x_1 = 1, x_2 = 1$, then $f_1 = -1, f_2 = 5$, and when $x_1 = 1, x_2 = -1$, then $f_1 = 3, f_2 = -1$. Find all coefficients of this system.

1.12.13. Assume first that in

$$\begin{bmatrix} 1 & -1 & 1 \\ 4 & 2 & 3 \\ 5 & 7 & 3 \end{bmatrix} x = f$$

$3f_1 - 2f_2 + f_3 = 0$ so that the system is consistent, and solve it for x_1, x_2, x_3. Then assume that $3f_1 - 2f_2 + f_3 = \delta \neq 0$, and solve successively the original system with one equation removed at a time,

$$\begin{bmatrix} 4 & 2 & 3 \\ 5 & 7 & 3 \end{bmatrix} x' = \begin{bmatrix} f_2 \\ f_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -1 & 1 \\ 5 & 7 & 3 \end{bmatrix} x'' = \begin{bmatrix} f_1 \\ f_3 \end{bmatrix}.$$

Then compute $x'_1 - x_1,\ x'_2 - x_2$ and $x''_1 - x_1,\ x''_2 - x_2$ in terms of δ.

1.13 Transposed systems

Much of theoretical linear algebra involves groping for concepts, terminology and forms to succinctly express its fundamental claims. We have already encountered the determinant, diagonal forms of type 0 and 1, the echelon and Hermite forms, and the rank of a linear system. Before we move on to an unusual though useful formulation of a fundamental theorem on linear equations, we need to carefully reconsider elementary operations and the linear combination of equations.

The purpose of elementary operations in the Gauss solution of a linear system is to progressively and systematically replace each equation of the system by a linear combination of the equation in question and another one in such a manner that the new replacing equation has a larger number of zero coefficients than the replaced equation. Successive application of these replacements in forward elimination and back substitution effectively replaces each equation of the system by a linear combination of all other equations with the intent of ending up with as many zero coefficients as possible within each replaced equation. Eventually the process removes all redundant $0 = 0$ equations from the system, and the system is brought to echelon form. At any stage of the solution process the *replaced* equation can be multiplied by any number other than zero. The solution of an $m \times n$ system is reduced thereby to a long sequence of solutions of one equation in one unknown.

A numerical example will bring some cogency to the discussion. Consider the replacement of an equation of the homogeneous system

$$\begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 7 & 12 & -13 & 3 & -8 \\ 5 & 3 & -2 & 3 & -1 \\ 8 & 23 & -27 & 2 & -17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad Ax = o \tag{1.82}$$

by the linear combination

$$\begin{aligned} &y_1(x_1 - 2x_2 + 3x_3 + x_4 + 2x_5) + y_2(7x_1 + 12x_2 - 13x_3 + 3x_4 - 8x_5) \\ &+y_3(5x_1 + 3x_2 - 2x_3 + 3x_4 - x_5) + y_4(8x_1 + 23x_2 - 27x_3 + 2x_4 - 17x_5) = 0. \end{aligned} \tag{1.83}$$

or

$$\begin{aligned} &(y_1 + 7y_2 + 5y_3 + 8y_4)x_1 + (-2y_1 + 12y_2 + 3y_3 + 23y_4)x_2 + (3y_1 - 13y_2 - 2y_3 - 27y_4)x_3 \\ &+(y_1 + 3y_2 + 3y_3 + 2y_4)x_4 + (2y_1 - 8y_2 - y_3 - 17y_4)x_5 = 0 \end{aligned} \tag{1.84}$$

of all other equations with weights or factors y_1, y_2, y_3, y_4. Suppose we choose $y_1 = -2, y_2 = 4, y_3 = 1, y_4 = -3$ and elect to replace the second equation of $Ax = o$ by eq. (1.84). This entails multiplication of equation 2 of $Ax = o$ by 4 and then the addition to it of equation 1 times -2, equation 3 times 1 and equation 4 times -3, all legitimate elementary operations. The choice $y_1 = 2, y_2 = -3, y_3 = 0, y_4 = -1$ permits the replacement of every equation of $Ax = o$ by equation (1.84), except for the third, *since the replaced equation may not be multiplied by zero*. Such a multiplication wipes out the replaced equation and does not constitute a legitimate elementary operation—it does not produce an equivalent system.

We are not interested in any y's, except for those that will possibly turn replacing eq. (1.84) into $0 = 0$. Setting the five coefficients of eq. (1.84) equal to zero results in the homogeneous system

$$\begin{bmatrix} 1 & 7 & 5 & 8 \\ -2 & 12 & 3 & 23 \\ 3 & -13 & -2 & -27 \\ 1 & 3 & 3 & 2 \\ 2 & -8 & -1 & -17 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1.85}$$

for the four weights or factors y_1, y_2, y_3, y_4.

All coefficients of system (1.85) are the same as those of $Ax = o$ in eq. (1.82), except that the ith *row* of coefficient matrix A is turned into the ith *column* in the coefficient matrix of system (1.85). Systems (1.85) and (1.82) have coefficient matrices that are the *transpose* of each other and we write $A^T y = o$ to indicate this fact. Coefficient matrix A with m rows and n columns is transposed into coefficient matrix A^T with n rows and m columns.

Nothing practical is gained by this comprehensive mode of equation replacements, however, since $A^T y = o$ still needs to be solved by forward elimination and back substitution. But we do it nonetheless for theoretical interest.

We routinely determine that $A^T y = o$ is of rank 2 and solve it as

$$y_1 = -\frac{3}{2}y_3 + \frac{5}{2}y_4,\ y_2 = -\frac{1}{2}y_3 - \frac{3}{2}y_4 \tag{1.86}$$

in which y_3 and y_4 are independent and y_1 and y_2 dependent. The fact that we may select some nonzero values for y_3 and y_4, say $y_4 = 1$ and $y_3 = 1$ to have nonzero $y_1 = 1, y_2 = -2$ implies that *any* one equation of system $Ax = o$ may be deleted. Actually, all we need to know is that y_3 and y_4 are arbitrary and *may* be assigned any nonzero value, so that either equation 3 or equation 4 is redundant.

We elect to delete equation 4 and the original system $Ax = o$ of four equations is reduced to

$$\begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 7 & 12 & -13 & 3 & -8 \\ 5 & 3 & -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ A'x = o \tag{1.87}$$

with only three remaining equations.

System $A'x = o$ may still harbor some more $0 = 0$ redundancies that should be discovered through another attempted replacement of one of its equations by linear combination (1.83) *excluding equation* 4. But this is exactly the same as seeking a solution to $A^T y = o$ under the restriction that $y_4 = 0$. Because y_4 and y_3 are arbitrary, we have in fact nontrivial solutions to $A^T y = o$ even if $y_4 = 0$. For instance $y_4 = 0, y_3 = 1, y_2 = -1/2, y_1 = -3/2$, and either equation 3,2 or 1 of $A'x = o$ may be deleted. We know that equation 3 of $A'x = o$, or $Ax = o$, may also be deleted by the mere fact that y_3 is arbitrary, and we elect to delete this equation so as to be left with

$$\begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 7 & 12 & -13 & 3 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ A''x = o. \tag{1.88}$$

No more redundancies happen to exist in $A''x = o$ since if $y_3 = y_4 = 0$, then also $y_1 = y_2 = 0$, and we bring $A''x = 0$ to the reduced echelon form

$$\begin{bmatrix} 1 & & 5/13 & 9/13 & 4/13 \\ & 1 & -17/13 & -2/13 & -11/13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1.89}$$

with all $0 = 0$ equations removed.

Conclusion: system $Ax = o$ includes two redundancies because the solution to $A^T y = o$ includes two arbitrary unknowns.

With this example completed we are ready to deal with some theorems on transposed systems.

Theorem 1.18. *Homogeneous systems $Ax = o$ and $A^T y = o$ have the same number of dependent unknowns—the two systems have the same rank.*

Proof. Let $Ax = o$ be a system of m equations in n unknowns so that $A^T y = o$ consists of n equations in m unknowns. If $A^T y = o$ has r dependent unknowns, is of rank r, and $m - r$ independent, then exactly $m - r$ equations of $Ax = o$ are redundant and may be deleted. We are left with $m - (m - r) = r$ equations and consequently with r leading 1's in the reduced echelon form of $Ax = o$, which is thus of rank r. Reversing the roles of $Ax = o$ and $A^T y = o$ completes the proof.

An $m \times n,\ n \geq m$, system of equations is of full rank m if it does not include any $0 = 0$ redundancies. Recall that if $m > n$, then the system includes at least $m - n$ redundant $0 = 0$ equations.

Corollary 1.19. *If square system $Ax = f$ is equivalent to a triangular system of type 1, then so is $A^T x = f'$, and if $Ax = f$ is equivalent to a triangular system of type 0, then so is $A^T x = f'$.*

Proof. According to Theorem 1.18 $Ax = o$ and $A^T x = o$ have the same rank. If one is of full rank, then so is the other, and if one is not of full rank, then neither is the other. End of proof.

Corollary 1.20. *A homogeneous $m \times n,\ n \geq m$, system of equations $Ax = o$ is of full rank m if and only if $A^T y = o$ has the unique trivial solution $y = o$.*

Proof. System $A^T y$ is with n equations in m unknowns. Since $n \geq m$, its highest rank is m. System $A^T y = o$ possesses a unique solution if and only if it is of rank m. End of proof.

In a somewhat different formulation the next theorem is called the *Fredholm alternative*.

Theorem 1.21. *A necessary and sufficient condition for the solution of the* $m \times n$ *system* $Ax = f$ *to exist, is for* f *to be such that for every solution* y *of the transposed* $n \times m$ *system* $A^T y = o$,

$$f_1 y_1 + f_2 y_2 + \cdots + f_m y_m = 0. \tag{1.90}$$

Proof. The condition is necessary. Suppose that $A^T y = o$ possesses a nontrivial solution with $y_i \neq 0$. Then it is legitimate to replace the ith equation of $Ax = f$ by the linear combination

$$\begin{gathered} y_1(Ax)_1 + y_2(Ax)_2 + \cdots + y_m(Ax)_m = y_1 f_1 + y_2 f_2 + \cdots y_m f_m \\ (Ax)_i = A_{i1}x_1 + A_{i2}x_2 + \cdots + A_{in}x_n \end{gathered} \tag{1.91}$$

which turns into

$$x_1(A^T y)_1 + x_2(A^T y)_2 + \cdots + x_n(A^T y)_n = y_1 f_1 + y_2 f_2 + \cdots + y_m f_m \tag{1.92}$$

in which $(A^T y)_i = 0$ for all i, and unless the right-hand side of the newly combined equation is zero, we face the contradiction $0 = 1$.

The condition is sufficient. If the solution to $A^T y = o$ contains k arbitrary unknowns, then by condition (1.90) exactly k equations of $Ax = f$ may be replaced by $0 = 0$ and deleted. The remaining $m - k$ equations are equivalent to an echelon form with a leading 1 coefficient in every equation and are therefore soluble. In particular, if $y = o$ is the only solution of $A^T y = o$, then $Ax = f$ is of full rank and therefore consistent. End of proof.

Corollary 1.22. $Ax \neq y,\ y \neq o$, *if* $A^T y = o$, *and conversely* $A^T y \neq o$ *if* $Ax = y,\ y \neq o$.

Proof. The corollary says that $Ax = y$ is insoluble if y is a nontrivial solution of $A^T y = o$. Conversely, if $Ax = y$, $(y \neq o)$, is soluble, then y cannot be a nontrivial solution of $A^T y = o$. For a proof set $y = f$ in eq. (1.90) and observe that for any real y

$$y_1^2 + y_2^2 + \cdots + y_m^2 > 0 \tag{1.93}$$

if at least one of the y's is nonzero. End of proof.

Theorem 1.23. *Let $Ax = o$ and $A'x = o$ have the same number of unknowns. A necessary and sufficient condition that every equation of one system can be expressed as a linear combination of the equations of the other system, is that the systems be equivalent.*

Proof. To shorten the writing without actually losing generality, we consider the two systems

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A_{11}' & A_{12}' & A_{13}' \\ A_{21}' & A_{22}' & A_{23}' \\ A_{31}' & A_{32}' & A_{33}' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{1.94}$$

To express the ith equation of $A'x = o$ in terms of the equations of $Ax = o$, we need y's so that

$$\begin{aligned} y_1(A_{11}x_1 + A_{12}x_2 + A_{13}x_3) + y_2(A_{21}x_1 + A_{22}x_2 + A_{23}x_3) \\ = A_{i1}'x_1 + A_{i2}'x_2 + A_{i3}'x_3. \end{aligned} \tag{1.95}$$

Equal coefficients for x_1, x_2, x_3 on both sides of eq. (1.95) are obtained with y's that satisfy

$$\begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \\ A_{13} & A_{23} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A_{i1}' \\ A_{i2}' \\ A_{i3}' \end{bmatrix} \tag{1.96}$$

where the coefficient matrix is A^T. If we show that eq. (1.96) is soluble, then we have proven that any equation of $A'x = o$ can be written as a linear combination of the equations of $Ax = o$. According to Theorem 1.21 a necessary and sufficient condition for the solubility of (1.96) is, that for every solution of $Ax = o$, $A_{i1}'x_1 + A_{i2}'x_2 + A_{i3}'x_3 = 0$, $i = 1, 2, 3$, or $A'x = o$. Reversing the roles of $Ax = o$ and $A'x = o$ we conclude that an arbitrary equation of $Ax = o$ can be written as a linear combination of the equations of $A'x = o$ if and only if every solution of $A'x = o$ is also a solution of $Ax = o$. Hence the linear combinations can be done on the one or the other if and only if the two systems have exactly the same solutions. End of proof.

Exercises.

1.13.1. System

$$\begin{bmatrix} 5 & -4 & 9 & -2 \\ 4 & -3 & 7 & -1 \\ 3 & -2 & 5 & 0 \\ 2 & -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

is insoluble. Which equations can be removed from it so as to leave a consistent system of highest rank?

1.13.2. Consider system

$$r_1 = 5x_1 - 2x_2 + 3x_3 + 3 = 0$$

$$r_2 = 4x_1 + 8x_2 + 3x_3 - 6 = 0$$

$$r_3 = 2x_1 - 4x_2 + x_3 + 4 = 0.$$

Can equation 2, $r_2 = 0$, be replaced by

$$\alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3 = 0, \ \ \alpha_2 \neq 0$$

so that it becomes $0 = 0$?

1.13.3. Solve $A^T y = o$ to determine the rank of

$$\begin{bmatrix} 1 & -2 & 1 & 3 \\ 1 & 8 & -5 & -5 \\ 2 & 1 & -1 & 2 \\ 1 & 2 & -1 & 3 \end{bmatrix} x = o, \ \ Ax = o.$$

Which of its equations are redundant?

1.13.4. Remove all $0 = 0$ redundancies from the system

$$\begin{bmatrix} -1 & 2 & 1 \\ 6 & -10 & -5 \\ 3 & -4 & -2 \\ 4 & -6 & -3 \end{bmatrix} x = o.$$

1.13.5. Remove the least number of equations from

$$\begin{bmatrix} 1 & -1 & 2 \\ 4 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} x = f$$

to have a consistent system for any f. Which are the disposable equations?

1.13.6. Consider the systems $Ax = o$ and its adjoint $A^T y = o$. The latter system is solved as $y_1 = y_3 + y_4, \ y_2 = 2y_3 + 2y_4, \ y_3 = y_3, \ y_4 = y_4$. Which equations in system $Ax = o$ are redundant?

1.14 Computer solution

Systems of linear equations that have their origin in mathematical physics and engineering are far larger than the 3×3, 4×4 or 5×5 systems encountered in classroom examples, and they are solved on the computer. In fact what most large computers in the service of science and industry do most of the time is solve large systems of linear equations describing some paradigmatic reality in one field of human inquiry or another by means of professionally prepared programs.

Gauss elimination for the solution of systems of equations is manifestly straightforward and is readily programmed. The principle of the solution is simple but its actual practical fulfilment can be bewildering to a degree unimagined by the student used to seeing only small systems with *integer* data.

Complications in the computer solution of large systems arise from three basic limitations of the computing machine: arithmetic execution time is limited, data storage space is limited, word length is limited.

Storage and execution time limitations cause *organizational* difficulties for the programmer faced with the need to store and retrieve efficiently vast amounts of numerical data that test the limits of computer capacity. We shall deal with this aspect of the solution procedure in chapter 3. In this section we concentrate on the *numerical* difficulties caused by the computer's ability to hold words of limited length only, or by its causing *round-off* errors in the arithmetic.

We can only offer hints and glimpses on this subject. Theoretical assessment of the damage that will be done or that has been done to the solution of a linear system by round-off errors is one of the most formidable challenges of computational linear algebra. It depends on certain deep properties of the linear system, on the algorithm used, and on the hardware. As matters stand now, round-off error analysis of linear systems is part exact mathematics, part statistical predictions, and part numerical observations. The theory of the solution of systems of equations is coherent and complete, but practice comes to suddenly obscure and muddle it.

To be able to handle very large as well as very small numbers with a fixed number of digits the computer resorts to *floating-point* arithmetic. The real number 36,754., for instance, is stored in a four digit machine that rounds off as .3675 10^5. But the number stored as .3675 10^5 could equally well have been any number between 36,745 and 36,755. There is an inherent uncertainty of ten units in this rounded off number. A number less than 10 added to 36,745 is not noticed in the computer representation of the sum. Similarly, subtraction of close numbers may cause a serious loss in the number of *significant digits* in their difference

$$.3676\ 10^5 - .3675\ 10^5\ = .0001\ 10^5 = .1 \times \times \times\ 10^2 \tag{1.97}$$

where $\times$ stands for an insignificant digit.

To observe the devastating effect of rounding-off in the subtraction of close numbers consider the simple

$$(1.23555 - 1.23444)x = (1.23678 - 1.23456) \tag{1.98}$$

that produces $x = 2$ in a 6-digit arithmetic, $x = 1.8333$ in a 5-digit arithmetic, and $x = 1$ in a 4-digit arithmetic. Long chains of a vast number of arithmetical operations complicate matters for systems of equations, rendering the subversive effect of round-off errors more insidious and elusive.

The largest positive number that when added to 1 is not recognized by the computer is the *round-off error unit* u of the arithmetic. For four digit arithmetic $u = 5.10^{-4}$. On a real computer typically $u = 10^{-7}$ in single precision and $u = 10^{-16}$ in double.

Finite word length means also that there are lower and upper limits to the size of the stored numbers. The computer operates in a discrete and finite world of real numbers. Common machines consider numbers lesser in magnitude than 10^{-78} as zero, and numbers larger than 10^{76} as $+\infty$. Such a computer will reject $10^{-78}x = 10^{-78}$ on the grounds of 0/0 ambiguity, and $10^{76}x = 10^{76}$ on ground of ∞/∞ ambiguity.

We must be sure, therefore, that systems of linear equations intended for computer solution are properly *scaled* so that none of the coefficients and right-hand sides is too small or too large for the machine to recognize.

We can also look at it this way: because of round-off, the associative law of addition does not strictly hold in floating-point arithmetic. If $|\epsilon| << 1$, then $1 + \epsilon - 1 + \epsilon$ is evaluated as $(1-1)+(\epsilon+\epsilon)=2\epsilon$, or as $(1+\epsilon)+(-1+\epsilon)=0$ depending on the order of evaluation.

As a result of rounding, mere *storage* of systems with fractional coefficients alters the data. In place of the original $Ax=f$ the computer is presented the slightly modified system $A'x'=f'$, and we actually never solve the system we intend to solve. The variation in each coefficient and right-hand side may seem small, in the sixth or seventh digit, but under certain circumstances the resulting change from the theoretical x to the theoretical x' can be considerable to an extent that no digit in x remains intact.

Small perturbation in the data of $Ax=f$ causing large fluctuations in the solution signal an inherent *numerical instability* of the system.

Moreover, elementary operations performed on a linear system produce a new system that is not strictly equivalent to the one modified. The total effect of the arithmetical errors in the Gauss solution can also be looked upon as a modification of the original system, and if the system is unstable, then even a small modification may cause serious errors in the solution.

Matters are compounded by the fact that if not executed carefully, the algorithm can inflict serious round-off errors on a solution to a system that is otherwise stable, as illustrated in the following example.

Consider the square 2×2 system

$$\begin{bmatrix}\epsilon & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix} \tag{1.99}$$

in which $|\epsilon|<1$ is a parameter. We are interested in seeing what a small ϵ can do to the solution of the system and how it affects the Gauss algorithm. Obviously, when $\epsilon=0, x_1=x_2=1$. Otherwise, if $\epsilon\neq -1$, then $x_1=x_2=1/(1+\epsilon)$, and x_1 and x_2 are only slightly affected by small changes in ϵ around $\epsilon=0$. Both x_1 and x_2 *continuously depend* upon ϵ near $\epsilon=0$ and the system is stable in this respect.

Even though it does not matter if $\epsilon=0$ or not for the exact solution, it matters greatly for the Gauss elimination procedure, for if $\epsilon\neq 0$, then, at least in theory, ϵ may be taken as first pivot, but if $\epsilon=0$ equations 1 and 2 must be interchanged. There is no continuous dependence on ϵ in this respect, but we are rather confronted with drastically different algorithmic eventualities in case $\epsilon\neq 0$ or $\epsilon=0$.

Say $\epsilon \neq 0$ and take it as first pivot to have

$$\begin{bmatrix} \epsilon & 1 \\ & -1 - 1/\epsilon \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/\epsilon \end{bmatrix} \tag{1.100}$$

by the elimination of x_1 from the second equation. Theoretically eq. (1.100) is equivalent to eq. (1.99), but not in practice. As ϵ tends to 0 a point is reached at which 1 becomes negligible compared to $1/\epsilon$ in floating-point arithmetic and it is computed that $x_2 = 1$, and consequently $x_1 = 0$, a solution greatly in error. Interchanging the first and second equations makes 1 first pivot, and a *good* solution is computed for (1.99) in floating-point arithmetic however small $|\epsilon|$ is.

System

$$\begin{bmatrix} 1+\epsilon & 1 \\ 1 & 1-\epsilon \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2+\epsilon \\ 2-\epsilon \end{bmatrix} \tag{1.101}$$

with $|\epsilon| < 1$ is different. When $\epsilon = 0$ the system is equivalent to an upper-triangular system of type zero possessing the multiple solutions $x_1 = \alpha, x_2 = 2 - \alpha$ for an arbitrary α. When $|\epsilon|$ is small but different from zero exact elimination produces

$$\begin{bmatrix} 1+\epsilon & 1 \\ & -\epsilon^2/(1+\epsilon) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2+\epsilon \\ -\epsilon^2/(1+\epsilon) \end{bmatrix} \tag{1.102}$$

and $x_1 = x_2 = 1$. The solution of system (1.101) does not depend continuously on ϵ as the system jumps from having many solutions at $\epsilon = 0$ to having a unique solution for $\epsilon \neq 0$.

There are no pivoting difficulties or dilemmas with system (1.101) even near $\epsilon = 0$, but in the presence of round-off errors computation of

$$((1-\epsilon) - 1/(1+\epsilon))x_2 = (2-\epsilon) - (2+\epsilon)/(1+\epsilon) \tag{1.103}$$

is subject to serious cancellation errors and loss of significance due to the subtraction of close numbers if $|\epsilon| << 1$, resulting in an $x_2 = 0/0$ ambiguity and a random x_2. But $x_1 + x_2$ is always near 2.

When $\epsilon = 0$ system

$$\begin{bmatrix} 1+\epsilon & 1 \\ 1 & 1-\epsilon \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \tag{1.104}$$

has multiple solutions; $x_1 + x_2 = 2$. When $\epsilon \neq 0,\ |\epsilon| < 1$, we compute $x_2 = -2/\epsilon,\ x_1 = 2/\epsilon$, and both x_1 and x_2 increase without bound as $\epsilon \to 0$.

Avoidance of small *initial* pivots is important, and this is how a computer program operates for systems with real coefficients. In addition to the coefficients and right-hand sides of the $n \times n$, $Ax = f$, the program is also given a relative round-off error tolerance $\epsilon = 2u$. Before elimination starts, the entire coefficient matrix is scanned for the largest entry in absolute value. If this happens to be A_{ij}, then $p_1 = A_{ij}$ is made first pivot by reordering both equations and unknown, and the list of unknowns is accordingly updated. Pivot p_1 is used to eliminate x_j from all the $n-1$ equations below the first and $Ax = f$ is changed into

$$\begin{bmatrix} p_1 & \times & \times & \times \\ & \times & \times & \times \\ & \times & \times & \times \\ & \times & \times & \times \end{bmatrix} \begin{bmatrix} x_j \\ x_k \\ \\ x_l \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \end{bmatrix}. \tag{1.105}$$

Next pivot p_2, $|p_2| \leq |p_1|$ is sought in the above manner among all the coefficients in the lower $(n-1) \times (n-1)$ portion of the coefficient matrix, and the procedure is repeated. Each time a pivot, say p_m, is selected $|p_m|/|p_1|$ is compared against ϵ. If $|p_m|/|p_1| = 0$, then the rest of the coefficients in the lower block are zero and the rank of the system is less than n. The program exits with a statement to that effect.

If $|p_m|/|p_1|$ is not zero but less than the prescribed ϵ, the program continues, and it may continue to the very end, but upon completion it issues a warning that the solution is suspect and that the system is possibly of rank less than n.

But this warning is only suggestive. Consider the unscaled, or imbalanced, system

$$\begin{bmatrix} 1 & -1 \\ -10^{-8} & 2\ 10^{-8} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10^{-8} \end{bmatrix}. \tag{1.106}$$

There is nothing numerically wrong with it, and the unique solution $x_1 = x_2 = 1$ is accurately computed. But upon encountering the small pivot 10^{-8} the program will issue the warning that the rank of the system is possibly 1. A small pivot that remains small is not as alarming as a small pivot created by the difference of two close numbers larger than 1.

Every numerical concept that depends on the determination of exact zero, such as rank and triangular forms of type 0 and 1, is tenuous for real numbers and floating point arithmetic. Rank can be machine-, or accuracy-dependent.

The strategy of permuting both *rows and columns* in the coefficient matrix in search of the largest current pivot is aptly called *complete pivoting*, as distinct from *partial pivoting*, in which only the current *column* is searched for the largest pivot. Complete pivoting is expensive—it may double the solution cost of the Gauss elimination algorithm—but it is safe for systems where nothing is known beforehand about the pivots.

Complete pivoting is not only practically but also theoretically interesting. For a rectangular system Gauss elimination with complete pivoting directly produces the echelon form of the system. One readily sees how

$$\begin{bmatrix} 1 & \times & \times & \times & \times \\ & 1 & \times & \times & \times \\ & & 1 & \times & \times \end{bmatrix} \begin{bmatrix} x_j \\ x_k \\ \\ \\ x_l \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ \times \end{bmatrix} \tag{1.107}$$

is set up by Gauss elimination with complete pivoting.

All along we have considered the linear system in all its generality. In subsequent chapters we shall see that many common sources of linear equations produce systems of special form and properties, for which elimination can be simplified and economized, and for which pivoting might not be essential.

We shall not say more on the theoretically perplexing issue of round-off errors except for this sweeping statement: that there is extensive numerical evidence accumulated to suggest that with complete pivoting the numerical errors occasioned by the Gauss solution of a real system are no larger than the errors initially introduced into the system by the rounded data. And this is comforting to know.

Exercises.

1.14.1. Bring the system

$$\begin{bmatrix} 2 & 1 & -1 \\ 2 & -2 & -3 \\ 4 & 2 & -1 \end{bmatrix} x = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} f$$

to the form

$$\begin{bmatrix} p_1 & \times & \times \\ & p_2 & \times \\ & & p_3 \end{bmatrix} x = \begin{bmatrix} 1 & & \\ \times & 1 & \\ \times & \times & 1 \end{bmatrix} f$$

by means of elementary operations that do not include multiplication of an equation by $\alpha \neq 1$. Show that in this case $|p_1 p_2 p_3| = 6$. Then show that the product of the pivots is ± 6 for *any* ordering of the equations.

1.14.2. Solve the system

$$\begin{bmatrix} 1 & -1 & \\ -1 & 2 & -1 \\ & -1 & 1+\epsilon^2 \end{bmatrix} x = \begin{bmatrix} 1 \\ \epsilon \\ -1 \end{bmatrix}$$

first with $\epsilon = 0$ then with $\epsilon \neq 0$. Find the dependence of x on ϵ and describe what happens as $\epsilon \to 0$.

1.14.3. Solve system

$$\begin{bmatrix} 3.26 & 2.16 \\ 2.16 & 1.43 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1.10 \\ -0.73 \end{bmatrix}$$

accurately, then solve it in a three digit precision, then in a four digit precision.

1.15 Tridiagonal systems

We here bring up the issue of the $n \times n$ system

$$\begin{bmatrix} a_1 & b_2 & & & \\ c_2 & a_2 & b_3 & & \\ & c_3 & a_3 & \ddots & \\ & & \ddots & \ddots & b_n \\ & & & c_n & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{bmatrix} \tag{1.108}$$

termed *tridiagonal*, not only because it is common and interesting, but to complement our discussion of numerical stability and computer round-off errors. Assuming nonzero pivots, system (1.108) is brought to the equivalent upper-triangular form

$$\begin{bmatrix} p_1 & b_2 & & & \\ & p_2 & b_3 & & \\ & & p_3 & \ddots & \\ & & & \ddots & b_n \\ & & & & p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} f_1{}' \\ f_2{}' \\ f_3{}' \\ \vdots \\ f_n{}' \end{bmatrix} \tag{1.109}$$

by means of the convenient *recursive* formula

$$\begin{aligned} & p_1 = a_1 \quad f_1{}' = f_1 \\ & p_k = a_k - b_k c_k / p_{k-1} \quad , \quad k = 2, 3, \ldots, n \\ & f_k{}' = f_k - f_{k-1}{}' c_k / p_{k-1} \end{aligned} \tag{1.110}$$

and is then solved as

$$x_n = f_n{}'/p_n \ , \ x_k = (f_k{}' - x_{k+1} b_{k+1})/p_k, \quad k = n-1, n-2, \ldots, 1. \tag{1.111}$$

Tridiagonal systems present a good opportunity to impress upon the student the fact that lack of a small pivot does not mean that the system is stable and not prone to round-off errors.

The $n \times n$ tridiagonal system

$$\begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \\ x_n \end{bmatrix} = \begin{bmatrix} 1/n \\ \\ \\ \\ \\ \end{bmatrix} \tag{1.112}$$

has its origin in computational mechanics. Forward elimination carries it into the form

$$\begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & 1 & -1 \\ & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \\ x_n \end{bmatrix} = \begin{bmatrix} 1/n \\ 1/n \\ \\ \\ 1/n \end{bmatrix} \tag{1.113}$$

in which every pivot is 1. Back substitution then yields the solution
$x_n = 1/n,\ x_{n-1} = 2/n, \ldots, x_1 = 1$.

It would be erroneous for us to jump to the conclusion that because all pivots are 1, system (1.112) is insensitive to small changes in the data. Actually, as n increases and the system grows in size it becomes more and more unstable, or *ill-conditioned*.

Tridiagonal system

$$\begin{bmatrix} \alpha & -\alpha & & & \\ -\alpha & \beta & -\alpha & & \\ & -\alpha & \beta & -\alpha & \\ & & -\alpha & \beta & -\alpha \\ & & & -\alpha & \beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \\ x_n \end{bmatrix} = \begin{bmatrix} 1/n \\ \\ \\ \\ \\ \end{bmatrix} \tag{1.114}$$

with $\alpha = 0.999$ and $\beta = 2.001$ appears to be only a slight variation of system (1.112) for which $x_1 = 1$ independently of n. A very accurate ($u = 10^{-32}$) solution of system (1.114) turned out the following results: For $n = 6,\ x_1 = 0.9744$, with the largest pivot being $p_6 = 1.015$; for $n = 12, x_1 = 0.8916$, with the largest pivot being $p_{12} = 1.030$; for $n = 24, x_1 = 0.6745$ with the largest pivot being $p_{24} = 1.048$. Quite serious deviations from $x_1 = 1$ if $\alpha = 1, \beta = 2$.

Above and beyond the theoretically murky affair of round-off error, what can we do to *practically* determine its total effect on the system we are entrusted with solving? Most obviously the system should be solved in various degrees of precision and the solutions compared. Our own computer has three such degrees of accuracy commonly referred to as *single precision* (SP), in which it holds 7 significant digits, *double precision* (DP), in which it holds 16 digits, and *extended precision* (XP), in which it holds 33 digits.

The $n \times n$ five-diagonal system

$$\begin{bmatrix} 2 & -3 & 1 & & & & \\ -3 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \\ \\ \\ x_n \end{bmatrix} = \begin{bmatrix} 1/n^2 \\ \\ \\ \\ \\ \\ \\ \end{bmatrix} \qquad (1.115)$$

is prone, we suspect, to round-off errors. To find out and estimate the accuracy of the solution it is stored and computer-solved for $n = 11$ in the three levels of accuracy; the three solutions are listed below (where .4179e01 means .4179 10^1.) Comparing the single and double precision solutions we observe

unknown	SP	DP	XP
1	.4179 e01	.418181818175 d01	.41818181818181818 q01
2	.4088 e0l	.40909090909084 d01	.40909090909090909 q01
3	.3914 e01	.39173553719002 d01	.39173553719008265 q01
4	.3666 e01	.36694214876027 d01	.36694214876033058 q01
5	.3353 e01	.33553719008259 d01	.33553719008264463 q01
6	.2981 e01	.29834710743797 d01	.29834710743801653 q01
7	.2560 e01	.25619834710740 d01	.25619834710743802 q01
8	.2097 e01	.20991735537187 d01	.20991735537190083 q01
9	.1602 e01	.16033057851237 d01	.16033057851239669 q01
10	.1082 e01	.10826446280990 d01	.10826446280991736 q01
11	.5450 e00	.54545454545445 d00	.54545454545454546 q00

that the first holds only two significant digits, or that $7-2=5$ digits are lost in the solution. This means that the double precision solution should be significant up to $16-5=11$ digits, which we see to be true by comparing it with the extended precision solution.

Answers.

Section 1.1

1.1.1. $x = -1, y = 1$; inconsistent; $x = t, y = 2t - 3$ for any t.

1.1.2. $f + f' = 0$.

1.1.3. $\alpha \neq 1$, No.

1.1.4. $f = -1,\ f' = -6,\ f^{''} = -5$.

1.1.5. $f = f' = f^{''} = 0$.

1.1.6. $\alpha\beta \neq\ -1$.

Section 1.2

1.2.4. $x = -2,\ x = x$.

1.2.5. $x = 7/4$.

1.2.7. $x = b$.

1.2.11. $x = 0, x = 1, x = -1$.

1.2.12. $x = 0, x = 1$.

1.2.29. $\Delta = \Delta' = 0$.

1.2.30. $\Delta = -2^3 \cdot 3^3$.

1.2.32. 2^4.

Section 1.6

1.6.2. $x_1 = x_2 = x_3 = x_4 = 0, x_5 = 2$.

1.6.3. Yes.

1.6.4. No.

1.6.5. $-\alpha + 3\beta + 3\gamma \neq 0$; $\alpha = -3\beta$, $\gamma = -3\beta$.

Section 1.7

1.7.2. $x_1 = -x_2 + x_3 + x_5$, $x_2 = x_2$, $x_3 = x_3$, $x_4 = -x_5$, $x_5 = x_5$, $x_6 = 0$.

1.7.3. 2.

1.7.4. 3.

1.7.5. 3.

1.7.10.

$$\begin{bmatrix} 1 & -1 & 2 & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} x = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \qquad \begin{gathered} x_1 = x_2 - 2x_3 - 2 \\ x_2 = x_2 \\ x_3 = x_3 \\ x_4 = 2 \\ x_5 = -1. \end{gathered}$$

1.7.11.

$$\begin{bmatrix} 1 & 2 & & -1 \\ & 0 & & \\ & & 1 & 3 \\ & & & 0 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 7 \\ 0 \end{bmatrix}, \qquad \begin{gathered} x_1 = -2x_2 + x_4 \\ x_2 = x_2 \\ x_3 = -3x_4 + 7 \\ x_4 = x_4. \end{gathered}$$

1.7.12.

$$\begin{bmatrix} 3 & & 4 & & 2 \\ & 3 & -5 & & -7 \\ & & 0 & & \\ & & & 1 & -1 \\ & & & & 0 \end{bmatrix} x = \begin{bmatrix} 9 \\ -9 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{gathered} x_1 = -\tfrac{4}{3}x_3 - \tfrac{2}{3}x_5 + 3 \\ x_2 = \tfrac{5}{3}x_3 + \tfrac{7}{3}x_5 - 3 \\ x_3 = x_3 \\ x_4 = x_5 \\ x_5 = x_5. \end{gathered}$$

1.7.14. Yes.

Section 1.8

1.8.1. $\alpha = 3$ or $\alpha = -2$.

Section 1.9

1.9.2.

$$\begin{bmatrix} 1 & 2 & & -3 \\ & & 1 & \\ & & & 0 \end{bmatrix} x = o$$

$x_1 = -2x_2 + 3x_4$, $x_2 = x_2$, $x_3 = 0$, $x_4 = x_4$.

1.9.3.

$$\begin{bmatrix} 1 & & 1 & 1 & 3 \\ & 1 & -1 & 1 & -1 \\ & & & & 0 \end{bmatrix} x = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$$

$x_1 = -x_3 - x_4 - 3x_5 + 5,\ x_2 = x_3 - x_4 + x_5 - 2,\ x_3 = x_3,\ x_4 = x_4,\ x_5 = x_5.$

Section 1.10

1.10.1. $x_1 = x_2 = 0.$

1.10.2.

$$\begin{bmatrix} 1 & & 1 \\ & 1 & \end{bmatrix} x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$x_1 = -x_3 + 2,\ x_2 = 1,\ x_3 = x_3.$

Section 1.12

1.12.3.

$$\begin{bmatrix} 6 & & \\ & -6 & \\ & & 2 \end{bmatrix} x = \begin{bmatrix} 1 & 7 & 1 \\ -1 & -1 & -1 \\ -1 & 3 & 1 \end{bmatrix} f.$$

1.12.5. $\alpha \neq -2,\ \alpha \neq 1.$

1.12.6. $f_1 + f_2 + f_3 = 0.$

1.12.7. $f_1 - 2f_2 + f_3 = 0,\ f_1 - f_2 + f_4 = 0.$

1.12.9. $B_{11} = 1/5,\ B_{12} = -2/5,\ B_{21} = 2/5,\ B_{22} = 1/5.$

1.12.11. $A_{11} = A_{12} = A_{21} = A_{22} = 1.$

1.12.12. $A_{11} = 1, A_{12} = -2, A_{21} = 2, A_{22} = 3.$

1.12.13. $x_1' - x_1 = -\delta/9,\ x_1'' - x_1 = \delta/12.$

Section 1.13

1.13.1. Equation 4.

1.13.2. Yes, $\alpha_1 = -2,\ \alpha_2 = 1,\ \alpha_3 = 3.$

1.13.3. One equation among equations 1,2,3.