9. Elastic equilibrium-finite elements

9.1 Perspectives

What makes linear algebra such a winning, vigorous chapter of applied mathematics is that it combines the purely theoretical with the very concrete. Most large scale linear algebraic computations and the theoretical questions they give rise to center on the solution and analysis of the discrete *equilibrium and evolution* equations of mathematical physics and theoretical engineering. A large part of what is on the agenda of applied linear algebra is mandated by the exigencies of practical computations and computer qualities and limitations.

In essence, discretization consists of the approximate replacement of a continuous physical behavior by a discrete one at many points in space and time. The result of such discretization is, as we have already seen in Chapter 3, vast systems of linear algebraic equations that are typically sparse, symmetric and positive semidefinite. Elastic equilibrium is the commonest computational problem that creates such large systems and we shall give this chapter a deliberate mechanical slant, first exploring the physical notion of force vectors.

In this book we shall not go so far as to actually discretize an elastic solid, a process which, if given full consideration, would carry us ever deeper into analysis. Instead we shall concentrate on a large scale, highly practical, elastic problem that is already discrete—that of the truss. A typical truss such as an antenna tower consists of thousands of rods interconnected at thousands of nodes, and its various states of elastic equilibrium under a variety of external loads are correspondingly described by thousands of equations with thousands of unknown node movements. The truss problem presents us with the perfect setting in which to accomplish the principal objective of this chapter, that of describing the finite element method for writing the system of equilibrium equations.

The second section of this chapter is devoted to rigid body kinematics as a preliminary to rigid body statics that is discussed in the following section. Section 4 introduces elasticity on a one degree of freedom rod problem, and section 5 extends the elastic equilibrium analysis to nonlinear behavior. We then pass from the one rod to the many, discuss some mathematical aspects of the truss, and culminate the chapter with the description of linear and nonlinear finite element methods.

9.2 Kinematics

Force is the cause of change in body velocity, and it is therefore important that we put the discussion of rigid body kinematics before that of equilibrium. Recall that rigid body movement is an isometry excluding reflection. During such motion the distance between any two points of the body remains unchanged.

We look first at the rigid motion of the plane.

Theorem 9.1 *The position of a plane rigid body is fixed by two distinct points on it, and that of a rigid space body by three noncolinear points on it.*

Proof. By contradiction. If A and B are two distinct points of the plane, and C any third point not on line AB, then the only distinct image of point C under isometry is point C' such that $AC = AC'$ and $BC = BC'$. See Fig.9.1. But the isometry is the forbidden reflection and hence point C' must be point C.

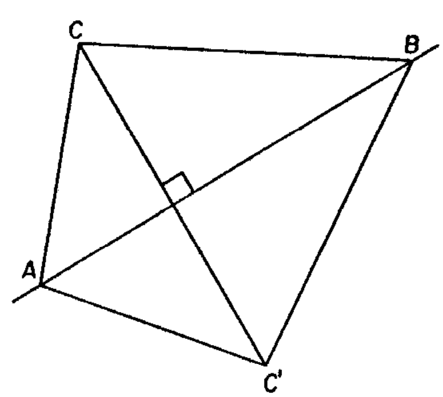


Fig. 9.1

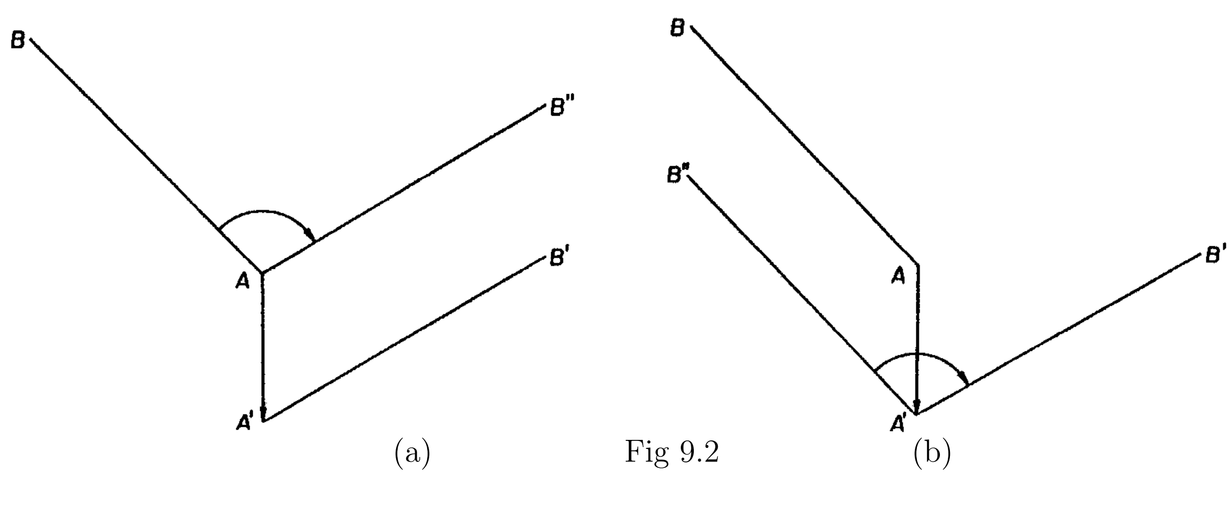
In space we consider triangle ABC and fourth point D not in the plane of the triangle. The only distinct point image D' of point D under isometry is a reflection in a plane mirror through points A, B, C, and hence point D must be point D'. End of proof.

We remain in the plane.

Theorem 9.2 *Plane rigid body motion consists of a translation followed by rotation around an arbitrary point of the plane by an angle that is independent of the point of rotation, or conversely.*

Proof. Two points fix the plane and it is enough that we consider the movement of two distinct arbitrary points A and B of the plane. Rigid body movement brings point A to point A', and point B to point B' so that distance $A'B'$ remains equal to distance AB. Isometry is a nonsingular mapping and to every pair of points A, B there corresponds only one image pair of points A', B' and vice versa.

The movement may be resolved into a rotation around point A to make AB'' parallel to $A'B'$, followed by translation $\vec{AA'}$ to bring point A upon point A' and point B'' upon point B' as in Fig.9.2(a). Otherwise, the same rigid body motion is accomplished by translation $\vec{AA'}$ followed by a rotation around point A' to bring point B'' to point B' as in Fig.9.2(b).



(a) Fig 9.2 (b)

Thus rigid body motion is accomplished by a translation followed by a rotation, or conversely.

We continue in a linear algebraic manner. Plane rigid body motion that sends arbitrary point P into point P' consists of rotation around some point C of the plane, that sends point P into point P'', followed by a translation to bring point P'' upon point P'. See Fig.9.3. With rotation matrix Q we write $\vec{CP''} = Q\vec{CP}$, and have from $\vec{OP} + \vec{PP''} + \vec{P''P'} = \vec{OP'}$ and $\vec{PP''} = \vec{CP''} - \vec{CP}$ that

$$p' = Qp + (I - Q)c + a = Qp + b, \; Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{9.1}$$

where p and p' are the position vectors of points P and P', respectively, where c is the position vector of center point C, and where $a = \vec{P''P'}$.

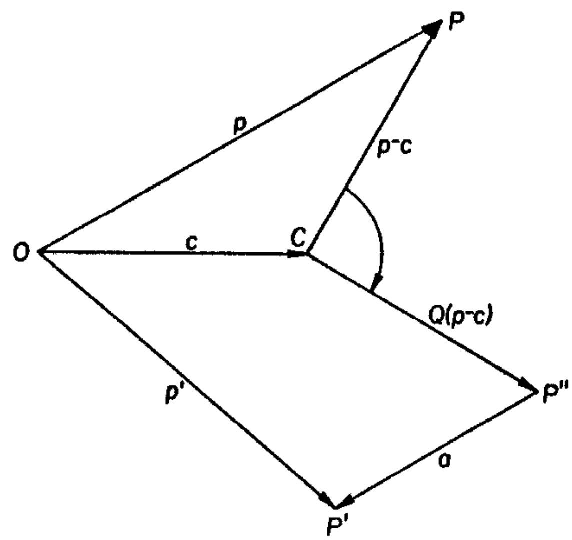


Fig. 9.3

For a different center of rotation C' we have $p' = Q'p + (I - Q')c' + a'$. Taking $p = o$ we establish that $(I-Q')c' + a' = (I-Q)c + a$, and by subtraction, that $p' - p' = o = (Q - Q')p$. Since this is true for arbitrary p of the plane it must happen that $Q - Q' = O$, $Q = Q'$, and $\cos\theta = \cos\theta'$, $\sin\theta = \sin\theta'$ and $\theta = \theta'$, so that $a' = a + (I - Q)(c - c')$. End of proof.

Pure rotation around point C turns point P into point P' so that $\vec{CP'} = Q\vec{CP}$,

$$p' = Qp + (I - Q)c, \; \det(I - Q) = 2(1 - \cos\theta) \tag{9.2}$$

and $I - Q$ is nonsingular for any $\theta \neq 0$. Comparing this with the general plane rigid body mapping $p' = Qp + b$ we see that $c = (I - Q)^{-1}b$ exists for any given b and $Q \neq I$. In other words, *every rigid body movement of the plane can be performed as one rotation around some displacement–dependent point C of the plane.*

Let p be the position vector of point $P(x, y)$ and p' the position vector of its image point $P'(x', y')$ under rigid body movement. According to Theorem 9.2 the motion can be described as an angle θ rotation around the origin followed by a translation of the origin, or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix}, p' = Qp + a \tag{9.3}$$

and the change of position of point P of the plane resulting from the rigid body motion is given by

$$p' - p = (Q - I)p + a. \tag{9.4}$$

Both in statics and dynamics we are often interested only in *differential*, or linearized, rigid body movements caused by the differential rotation $d\theta$ and differential translation $da = [du\ dv]^T$. We write in this case $p' - p = [dx\ dy]^T$ and have with $\cos(d\theta) = 1$ and $\sin(d\theta) = d\theta$, that

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = d\theta \begin{bmatrix} -y \\ x \end{bmatrix} + du \begin{bmatrix} 1 \\ 0 \end{bmatrix} + dv \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{9.5}$$

Considering the motion to be a continuous function of a parameter, say time t, we divide the above vector equation by dt and get

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \dot{\theta} \begin{bmatrix} -y \\ x \end{bmatrix} + \dot{u} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \dot{v} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{9.6}$$

where $\dot{x} = dx/dt, \dot{y} = dy/dt, \dot{\theta} = d\theta/dt$, $\dot{u} = du/dt$ and $\dot{v} = dv/dt$. Function $\dot{\theta} = \dot{\theta}(t)$ is the momentary *angular velocity* of the plane, while $\dot{u}$ and $\dot{v}$ are rectilinear velocities parallel to the x and y coordinates, respectively.

If $\dot{u} = \dot{v} = 0$, that is, if the origin is fixed, then velocity vector $\dot{p} = [\dot{x}\ \dot{y}]^T = \dot{\theta}[-y\ x]^T$ of point $P(x, y)$ is orthogonal to position vector $p = [x\ y]^T$ and is of magnitude $\|\dot{p}\| = |\dot{\theta}|(x^2 + y^2)^{1/2}$. The plane rotates then as a whole around the origin.

Let $P_1(x_1, y_1), P_2(x_2, y_2), \ldots, P_n(x_n, y_n)$ be n distinct points on the rigidly moving plane. Their velocity $\dot{p} = [\dot{p}_1^T\ \dot{p}_2^T\ \ldots\ \dot{p}_n^T]^T, \dot{p}_i = [\dot{x}_i\ \dot{y}_i]^T$ is given by

$$\dot{p} = \dot{\theta}r_1 + \dot{u}r_2 + \dot{v}r_3 \tag{9.7}$$

in which

$$r_1 = \begin{bmatrix} -y_1 \\ x_1 \\ -y_2 \\ x_2 \\ \vdots \\ -y_n \\ x_n \end{bmatrix}, \; r_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \; r_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{9.8}$$

are the three *rigid body modes* of the n point system.

To prove that for a system of two points or more, rigid body modes r_1, r_2, r_3 are linearly independent we write $\alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3 = o$ and have from it that $(x_2 - x_1)\alpha_1 = 0$ and $(y_2 - y_1)\alpha_1 = 0$. Points $A_1(x_1, y_1)$ and $A_2(x_2, y_2)$ being distinct, $x_2 - x_1$ and $y_2 - y_1$ cannot vanish at once and $\alpha_1 = 0$. Consequently $\alpha_2 = 0$ and $\alpha_3 = 0$.

For one point

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = d\theta \begin{bmatrix} -y_1 \\ x_1 \end{bmatrix} + du \begin{bmatrix} 1 \\ 0 \end{bmatrix} + dv \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{9.9}$$

and a nontrivial rigid body motion exists, namely rotation around $A_1(x_1, y_1)$, that leaves the point unmoved. Not so for two points. No *infinitesimal* plane rigid body movement consisting of $d\theta, du, dv$ exists that leaves two distinct points of the plane at rest. Allowing for *large* or *finite* movements, a full turn around one of the two points does return the second point back to its original position.

In case of pure rotation around point $C(x_0, y_0)$

$$p' - p = (Q - I)(p - c) \tag{9.10}$$

and

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \dot{\theta} \begin{bmatrix} -(y - y_0) \\ x - x_0 \end{bmatrix} \tag{9.11}$$

or

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \dot{\theta} \begin{bmatrix} -y \\ x \end{bmatrix} + y_0\dot{\theta} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - x_0\dot{\theta} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{9.12}$$

implying rotation around the origin followed by the instantaneous $\dot{\theta}[y_0 - x_0]^T$.

Any column elementary operation on the three rigid body mode vectors r_1, r_2, r_3 produces another set of three linearly independent modes. Write $R = [r_1 \; r_2 \; r_3]$, and let $T = T(3 \times 3)$

be a nonsingular matrix. Then $[r_1' \; r_2' \; r_3'] = RT$ contains a new set of three rigid body modes. If $C(x_0, y_0), C'(x_0', y_0'), C''(x_0'', y_0'')$ are three noncolinear points of the plane, then

$$T = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ y_0 & y_0' & y_0'' \\ x_0 & x_0' & x_0'' \end{bmatrix} \tag{9.13}$$

is nonsingular (Prove!) and

$$[r_1' \; r_2' \; r_3'] = \begin{bmatrix} -y_1 & 1 & 0 \\ x_1 & 0 & 1 \\ -y_2 & 1 & 0 \\ x_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ y_0 & y_0' & y_0'' \\ -x_0 & -x_0' & -x_0'' \end{bmatrix} \tag{9.14}$$

contains a set of three rigid body modes. Each mode is an instantaneous rotation around one of the points C, C', C'' and hence *any infinitesimal rigid body movement of the plane can be carried out as three infinitesimal rotations around three noncolinear, arbitrary, motion independent, points of the plane.*

The reader should have no difficulty showing that the general *finite* rigid body movement of the plane cannot be carried out as three rotations around three arbitrary points of the plane.

From the plane we move on to space.

Theorem 9.3 *Any rigid body motion of space consists of a translation followed by a unique rotation around an axis through an arbitrary point of space, or conversely.*

Proof. Since three noncolinear points fix a rigid body it is enough that we consider arbitrary, nondegenerate triangle ABC in space with its congruent image $A'B'C'$ under rigid body motion, as in Fig 9.4. The motion is reversible and for any arbitrary point A there corresponds one image point A' and conversely. Translation brings point A upon point A', and there is obviously a rotation around an axis through point A' that subsequently sends point C to point C', and a following rotation around $A'C'$ that finally brings point B to point B'. We know from Chapter 4 that the results of two (finite) rotations around axes through a common point is a rotation around an axis through the same point. Hence any rigid body motion can be carried out by a translation followed by a rotation around an axis through any point of space. The reader should easily prove the converse.

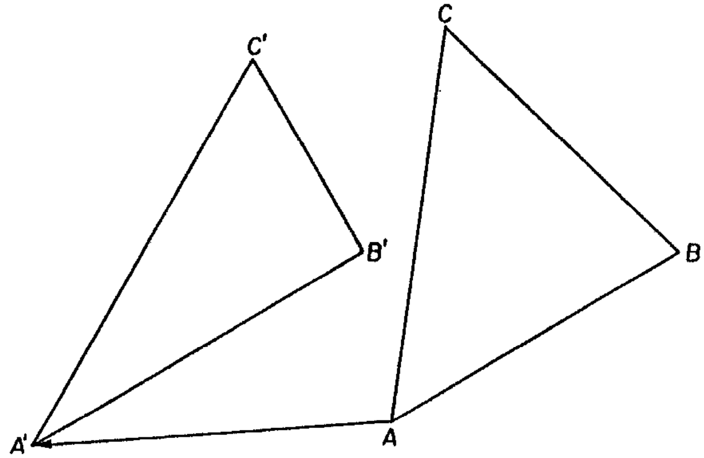


Fig. 9.4

From Section 4.9 we recall that the space rotation matrix for an angle θ turn around axis $n = [n_1 \ n_2 \ n_3]^T$ is

$$Q = I\cos\theta + (1-\cos\theta)nn^T + N\sin\theta, \ N = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \tag{9.15}$$

and as for the plane, also here, space rigid body motion that sends arbitrary point $P(x, y, z)$ into point $P'(x', y', z')$ is always written as $p' = Qp + b$, $b = (I - Q)c + a$, where c is the position vector of point C on the axis of rotation, and where $a = [u \ v \ w]^T$ is a translation. For another choice of point C and axis $n, p' = Q'p + b'$, and choosing $p = o$ we get that $b = b'$, and by subtraction that $p' - p' = o = (Q - Q')p$. Since this happens for arbitrary p it results that $Q = Q'$. Space rotation matrix Q is not changed if θ is replaced by $-\theta$ and n by $-n$, and we discount this possibility. Otherwise, from $Q = Q'$ and $Qn = n$ we have that $Q'n = n$ and the axis of rotation for Q' is colinear with that of Q'. From $Q_{11} + Q_{22} + Q_{33} = Q'_{11} + Q'_{22} + Q'_{33}$ we obtain that $\cos\theta = \cos\theta'$, and from $N\sin\theta = N\sin\theta'$, that $\sin\theta = \sin\theta'$. Hence $\theta = \theta'$. End of proof.

For a differential $d\theta, Q = I + Nd\theta$, and for an infinitesimal movement, $dp = p' - p = d\theta N(p - c) + da$. Small pure rotation causes the small movement $dp = d\theta N(p - c)$, and since $N(p - c)$ is orthogonal to both $p - c$ and n, movement dp due to rotation $d\theta$ is on a plane orthogonal to n and in direction orthogonal to $p - c$.

Let $c = o$ so that $dp = d\theta N p + da$. After division by dt we get

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \dot{\theta} n_1 \begin{bmatrix} 0 \\ -z \\ y \end{bmatrix} + \dot{\theta} n_2 \begin{bmatrix} z \\ 0 \\ -x \end{bmatrix} + \dot{\theta} n_3 \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix} + \dot{u} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \dot{v} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \dot{w} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{9.16}$$

which means that the arbitrary instantaneous rigid body movement of space resolves into the sequence of three independent rotations $\dot{\theta}_1 = \dot{\theta} n_1, \dot{\theta}_2 = \dot{\theta} n_2, \dot{\theta}_3 = \dot{\theta} n_3, \dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2 = \dot{\theta}^2$, around the x, y, z axes followed by the three independent slidings $\dot{u}, \dot{v}, \dot{w}$ along the same axes. The rigid plane has three degrees of freedom—a rotation and two translations—while rigid space has six degrees of freedom—three rotations and three translations.

A system of n points on a rigid body has six rigid body modes, which for $n = 3$ are the six columns of

$$[r_1 \; r_2 \; r_3 \; r_4 \; r_5 \; r_6] = \begin{bmatrix} 0 & z_1 & -y_1 & 1 & 0 & 0 \\ -z_1 & 0 & x_1 & 0 & 1 & 0 \\ y_1 & -x_1 & 0 & 0 & 0 & 1 \\ 0 & z_2 & -y_2 & 1 & 0 & 0 \\ -z_2 & 0 & x_2 & 0 & 1 & 0 \\ y_2 & -x_2 & 0 & 0 & 0 & 1 \\ 0 & z_3 & -y_3 & 1 & 0 & 0 \\ -z_3 & 0 & x_3 & 0 & 1 & 0 \\ y_3 & -x_3 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{9.17}$$

The six rigid body modes of a system of n points of which at least three are noncolinear are linearly independent. Rotation around an axis passing through two points leaves the points immovable. But no infinitesimal space rigid body movement exists that leaves three noncolinear points fixed.

Theorem 9.4 *The six space rigid body modes $r_1, r_2, r_3, r_4, r_5, r_6$ are linearly independent.*

Proof. We write $\alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3 + \alpha_4 r_4 + \alpha_5 r_5 + \alpha_6 r_6 = o$ and have after some elementary row operations that

$$\begin{bmatrix} & z_1 & -y_1 & 1 & & \\ -z_1 & & x_1 & & 1 & \\ y_1 & -x_1 & & & & 1 \\ & z_2 - z_1 & -y_2 + y_1 & & & \\ -z_2 + z_1 & & x_2 - x_1 & & & \\ y_2 - y_1 & -x_2 + x_1 & & & & \\ & z_3 - z_1 & -y_3 + y_1 & & & \\ -z_3 + z_1 & & x_3 - x_1 & & & \\ y_3 - y_1 & -x_3 + x_1 & & & & \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix} = o \tag{9.18}$$

from which we pick out the three systems

$$\begin{bmatrix} z_2 - z_1 & -y_2 + y_1 \\ z_3 - z_1 & -y_3 + y_1 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \alpha_3 \end{bmatrix} = o, \begin{bmatrix} -z_2 + z_1 & x_2 - x_1 \\ -z_3 + z_1 & x_3 - x_1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_3 \end{bmatrix} = o,$$

$$\begin{bmatrix} y_2 - y_1 & -x_2 + x_1 \\ y_3 - y_1 & -x_3 + x_1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = o. \tag{9.19}$$

The determinants of the three 2×2 systems are

$$\delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}, \ \delta_2 = - \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \end{vmatrix}, \ \delta_3 = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \tag{9.20}$$

respectively, recognized to be proportional to the projected areas of triangle $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$ upon the yz plane, the xz plane, and the xy plane, respectively. For a nondegenerate triangle at least one projected area is not zero. Assume $\delta_1 \neq 0$. Then the first of the three systems is nonsingular possessing the trivial solution $\alpha_2 = \alpha_3 = 0$ only. The two remaining equations leave us with

$$(-z_2 + z_1)\alpha_1 = 0, (-z_3 + z_1)\alpha_1 = 0, (y_2 - y_1)\alpha_1 = 0, (y_3 - y_1)\alpha_1 = 0 \tag{9.21}$$

and since points P_1, P_2, P_3 are noncolinear at least one of the coefficients must be different from zero, and $\alpha_1 = 0$. Consequently $\alpha_4 = \alpha_5 = \alpha_6 = 0$. End of proof.

Movement of a rigid body may not be entirely free, but restricted or *constrained*. Equality constraints that fix some points of the solid or that limit its movement to a curve or surface are *bilateral* constraints. Typical bilateral constraints are rotation around an axis, and sliding on a smooth frictionless table. Inequality constraints are *unilateral*. Typical to these constraints are wall limits on rectilinear movements and angle stops on rotations.

We are interested here in bilateral or equality constraints only, and as we are dealing with small motions mainly we assume the constraints linear, of the form

$$q_1^T dp = 0, \ q_2^T dp = 0, \ldots, q_m^T dp = 0 \tag{9.22}$$

for m given q vectors.

For instance, if a plane rigid body with the two distinct points $P_1(x_1, y_1), P_2(x_2, y_2)$ on it is restricted to rotation around the origin, then

$$q_1 = \begin{bmatrix} x_1 \\ y_1 \\ 0 \\ 0 \end{bmatrix}, \ q_2 = \begin{bmatrix} 0 \\ 0 \\ x_2 \\ y_2 \end{bmatrix}, \ dp = d\theta \begin{bmatrix} -y_1 \\ x_1 \\ -y_2 \\ x_2 \end{bmatrix} + du \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + dv \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \tag{9.23}$$

and the conditions $q_1^T dp = q_2^T dp = 0$ amount to

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{9.24}$$

and $du = dv = 0$ if $x_1 y_2 - x_2 y_1 = 0$.

exercises

9.2.1. Rigid body motion of the plane affects $A(0,0) \rightarrow A'(4,0),\ B(3,4) \rightarrow B'(9,0)$. Find point C around which the plane may be rotated to achieve this transformation. What is the angle of rotation?

9.2.2. What is the resultant instantaneous motion of a plane moving with angular speeds $\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3$ around points $P_1(0,0), P_2(1,0), P_3(0,1)$, respectively?

9.2.3. Show that every space rigid body motion is the composition of two rotations.

9.2.4. A twist consists of a translation followed by rotation around an axis parallel to the translation. Prove that any space rigid body motion consists of a unique twist.

9.2.5. Take the three plane rigid body modes r_1, r_2, r_3 of eq.(9.8) and replace r_1 by $r_1' = r_1 + \alpha_2 r_2 + \alpha_3 r_3$ so that r_1' is orthogonal to r_2 and r_3. Give r_3' a geometrical interpretation. Do the same to r_1, r_2, r_3 of the six space rigid body modes $r_1, r_2, r_3, r_4, r_5, r_6$ of eq.(9.17).

9.3 Statics

Consider rigid body B with n points $P_1, P_2, \ldots, P_n$ of it at which the n forces $F_1, F_2, \ldots,$ F_n act. To accommodate the notational tradition of classical mechanics we shall use capital Roman letters to mark the directed segments that represent force, but shall follow the linear algebraic custom in using lower case letters to denote the force when written out in terms of its three components. So, referring to the Cartesian coordinate system we describe the ith force as $f_i = [X_i\ Y_i\ Z_i]^T$. This is how the xyz components of the force appear in the classical literature.

Force F_i is geometrically represented by a directed segment, *but it is not entirely free.* For reasons soon to become clear the force may not be freely translated in space from point to point, except for sliding along its axis through point P_i. We call the axis of a force the line colinear with it through the point of application. Corresponding to point numbering system $1, 2, \ldots, n$ we write the total load vector $f = [f_1^T \; f_2^T \; \ldots \; f_n^T]^T$.

Vector f need not be constant. Movement of point P_i resulting from slight changes of position of body B may induce variations in the forces acting on it. The forces may well be position dependent so that

$$X_i = X_i(x, y, z), \; Y_i = Y_i(x, y, z), \; Z_i = Z_i(x, y, z) \tag{9.25}$$

with dependence on x, y, z that can be gradual or abrupt. In the case of a force due to a stretched spring attached to point P_i a small movement of the point that further extends the spring only slightly causes only a slight change in the force. On the other hand, if the force is due to a force transmitting contact with another rigid body, then any movement, however small, that breaks the contact causes the force to suddenly disappear. Friction forces that spring to life at an attempt to slide one rough body over another are also of the sudden kind.

Position dependent forces are the rule in applied mechanics and we shall accept them provided that they change continuously with position and have partial derivatives that also change continuously with position so as to admit the linearizations

$$\begin{aligned} X_i(x,y,z) &= X_i(x_i,y_i,z_i) + (\frac{\partial X}{\partial x})_i dx + (\frac{\partial X}{\partial y})_i dy + (\frac{\partial X}{\partial z})_i dz \\ Y_i(x,y,z) &= Y_i(x_i,y_i,z_i) + (\frac{\partial Y}{\partial x})_i dx + (\frac{\partial Y}{\partial y})_i dy + (\frac{\partial Y}{\partial z})_i dz \\ Z_i(x,y,z) &= Z_i(x_i,y_i,z_i) + (\frac{\partial Z}{\partial x})_i dx + (\frac{\partial Z}{\partial y})_i dy + (\frac{\partial Z}{\partial z})_i dz \end{aligned} \tag{9.26}$$

around their point of application.

Now that we know with what forces we have to deal and what differential displacements are, we are ready to introduce the

Definition. *Let* $f = [f_1^T \; f_2^T \; \ldots \; f_n^T]^T$ *be the total load vector of the forces acting at points* $P_1, P_2, \ldots, P_n$ *of rigid body* B. *Load vector* f *is allowed to have position dependent components as long as they are continuous and have continuous first derivatives at their points of application. Let also* $dp = [dp_1^T \; dp_2^T \ldots \; dp_n^T]^T$ *be an arbitrary, constraint abiding,* *differential movement of body B, with the n forced points on it. Then, scalar differential $f^T dp$ is the virtual work of the n forces resulting from virtual movement dp.*

Because of the high degree of continuity requirements on the forces and because of the small, differential, or virtual, nature of the displacements, the virtual work is of unchanged forces at their original point of application. The cardinal law of statics stipulates that a rigid body is in equilibrium if and only if the virtual work of the forces acting on it is zero for any admissible, constraint obliging, differential rigid body movement. In other words, the system of forces is in equilibrium if and only if load vector f is orthogonal to any differential displacement dp. If the load vector does happen to have a nonzero component in the direction of some dp, then motion will be initiated by the forces in that direction. No motion is possible that violates the geometrical constraints imposed on the body.

In case of free plane motion, $dp = d\theta r_1 + du r_2 + dv r_3$, where r_1, r_2, r_3 are the three rigid body modes of the n point system; and $f^T dp = 0$ can happen for any $d\theta, du, dv$, only if $f^T r_1 = 0, f^T r_2 = 0, f^T r_3 = 0$. Plane rigid body is in equilibrium if and only if load vector f is orthogonal to any of its three linearly independent rigid body modes.

Orthogonality conditions $f^T r_1 = 0, f^T r_2 = 0, f^T r_3 = 0$ produce, for the three modes in eq. (9.8), the three equations of equilibrium

$$M_1 + M_2 + \ldots + M_n = 0 \quad X_1 + X_2 + \ldots + X_n = 0 \quad Y_1 + Y_2 + \ldots + Y_n = 0 \tag{9.27}$$

where

$$M_i = -y_i X_i + x_i Y_i \tag{9.28}$$

is the moment about the origin of force $f_i = [X_i \ Y_i]^T$ acting at point $P_i(x_i, y_i)$. To better see the geometrical meaning of the moment look at Fig.9.5. Writing the force and position vectors as

$$f_i = \|f_i\| \, [\cos\alpha \ \ \sin\alpha]^T \quad \text{and} \quad p_i = \|p_i\| \, [\cos\beta \ \ \sin\beta]^T \tag{9.29}$$

we arrive at

$$M_i = \|f_i\| \, \|p_i\| \sin(\alpha - \beta) \tag{9.30}$$

and notice that $\delta = \|p_i\| \sin |\alpha - \beta|$ is the distance of the axis of force f_i from the origin. If $\alpha > \beta$, if the force tends to rotate the plane counterclockwise, then $M_i > 0$, but if $\alpha < \beta$, if the force tends to rotate the plane clockwise, then $M_i < 0$. We actually want to consider the moment of plane force F_i around any point C of the plane as vector $m_i = M_i[0\ 0\ 1]^T$, with $|M_i|$ being the product of the force magnitude by the distance of its axis from point C, and a sign convention as for the origin.

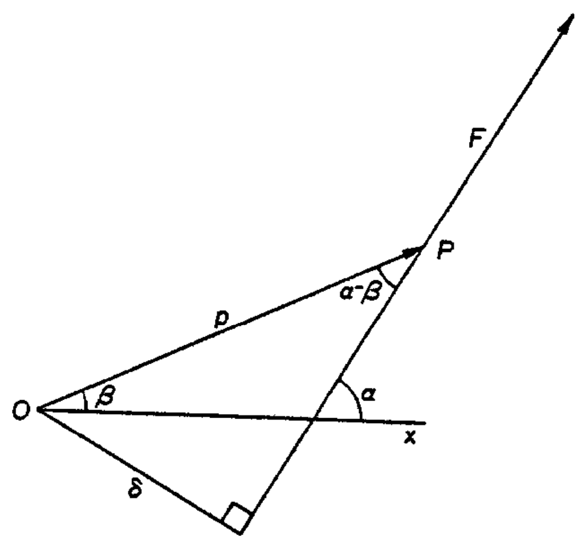


Fig 9.5

The second and third equations of equilibrium require that the vector sum of the forces, considered free in the plane, be zero. These two equations are indifferent as to where the forces act. The first equation of equilibrium asks that the moment vector sum of all the forces about the origin be zero. For this equation, sense, magnitude and axis distance from the origin are important, but not where the force is found along its axis.

Plane rigid body motion that led to the three equations of equilibrium was specifically written as a rotation around the origin followed by sliding along the coordinate axes. It could as well have been expressed as a rotation around any other single point of the plane followed by two slidings along any noncolinear axes of the plane. Indeed, if rotation is around point $C(x_0, y_0)$, then $dp = d\theta[-y + y_0\ x - x_0]^T + db$,

$$M_i = -(y_i - y_0)X_i + (x_i - x_0)Y_i \tag{9.31}$$

and the moments are taken now about point C. Generally, the three equations of equilibrium of the body confined to plane movement are the vanishing of the moment sum of all forces around any arbitrary point of the plane, and zero sums for all force components along any two noncolinear axes of the plane.

In space the situation is entirely analogous. An arbitrary differential free rigid body movement is given here by $dp = d\theta_1 r_1 + d\theta_2 r_2 + d\theta_3 r_3 + du r_4 + dv r_5 + dw r_6$ for independent differentials $d\theta_1, d\theta_2, d\theta_3, du, dv, dw$. Virtual work $f^T dp$ is zero for any dp if and only if load vector f is orthogonal to all six rigid body modes $r_1, r_2, r_3, r_4, r_5, r_6$ of the body with the $n \geq 3$ points on it. From the six modes of eq. (9.17) we obtain the six equations of equilibrium

$$M_1' + M_2' + \ldots + M_n' = 0 \quad M_1'' + M_2'' + \ldots + M_n'' = 0 \quad M_1''' + M_2''' + \ldots + M_n''' = 0$$

$$X_1 + X_2 + \ldots + X_n = 0 \quad Y_1 + Y_2 + \ldots + Y_n = 0 \quad Z_1 + Z_2 + \ldots + Z_n = 0 \tag{9.32}$$

where

$$m_i = \begin{bmatrix} M_i' \\ M_i'' \\ M_i''' \end{bmatrix} = \begin{bmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix} \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix}, \ m_i = N_i f_i \tag{9.33}$$

is the moment about the origin of force $f_i = [X_i \ Y_i \ Z_i]^T$ acting at a point with position vector $p_i = [x_i \ y_i \ z_i]^T$.

Vector m_i is readily seen to be orthogonal to both p_i and f_i. Writing $f_i = \|f_i\| u, \ u^T u = 1$, we obtain that

$$\|m_i\| = \delta \|f_i\|, \ \delta = (u^T \ N_i^T \ N_i \ u)^{1/2}, N_i^T N_i = \begin{bmatrix} z_i^2 + y_i^2 & -x_i y_i & -x_i z_i \\ -y_i x_i & x_i^2 + z_i^2 & -y_i z_i \\ -z_i x_i & -z_i y_i & y_i^2 + x_i^2 \end{bmatrix} \tag{9.34}$$

and ascertain through comparison with $\delta = \sqrt{p^T p - (u^T p)^2}$ that δ is the distance from the origin of the axis of force f_i acting at the point with position vector p.

The first three equations of equilibrium for the free rigid body in space require that the vector sum of the moments about the origin of all forces be zero, while the last three equations demand that the vector sum of all forces be zero. For the last three equations the forces may be considered free, while for the first three, the forces may only slide along their axes.

Differential rigid body motion carried out with rotation around an axis through point C of space is written as $dp = d\theta N(p - c) + da$ and the first three equations of equilibrium for the moment components obtained from $f^T dp = 0$ with this dp are for moments about point C rather than about the origin.

In sum:

A free rigid body with n forces applied to n of its points is in equilibrium if and only if the virtual work of the applied total load vector f is zero for any differential rigid body movement; or if and only if the total load vector f of the forces acting on the body is orthogonal to any six linearly independent rigid body modes of the n point system; or if and only if the vector sum of the moments of all acting forces about any one point of space is zero, and the vector sum of all forces acting on the body is zero. The latter is equivalent to the condition that the component sum of all forces along three noncoplanar lines vanish.

exercises

9.3.1. Show that if a system of three forces is in equilibrium, then the forces are on one plane and have concurrent axes. Consider also intersection at infinity.

9.3.2. Show that a necessary and sufficient condition that a system of forces acting on a rigid body be in equilibrium is that the moments of these forces about three noncolinear points vanish.

9.3.3. Suppose that a system of forces $f_1, f_2, \ldots, f_n$ acts on a rigid body to produce moments $m_1, m_2, \ldots, m_n$ about some point of space. Show that

$$(f_1 + f_2 + \ldots + f_n)^T \ (m_1 + m_2 + \ldots + m_n)$$

is independent of the point about which the moments were taken.

9.4 Elastic systems of one degree of freedom

We open discussion on the equilibrium and stability of discrete elastic systems by first considering the simplest, yet most basic, component of such systems–that of the thin rod under axial forces. What we call rod, or bar, is an ideally slender cylindrical piece of elastic solid structurally used to transmit axial forces, both *tensile* and *compressive*. Ropes and strings can transmit tensile forces but buckle under compressive forces. Being made of an elastic material the rod shortens and elongates by the action of the axial force, but the change in length is realistically so small that we may assume at first approximation that the size of the rod is practically unchanged by the load.

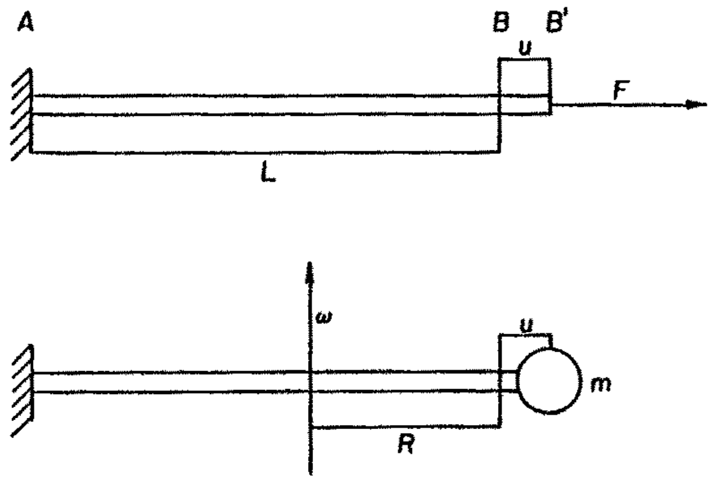


Fig 9.6

Consider the rod in Fig.9.6, fixed to a wall at end point A and acted upon by a perfectly axial force F at other end point B. Application of the force causes the rod to extend and as a result point B moves distance u to point B', at which point the elastic restoring force equals that of the applied load. Experiments suggest that for common elastic (those returning to their original shape upon removal of the applied loads) materials, and loads well below the rupture level, the elastic displacement u is symmetrically proportional to the applied force so that at equilibrium $F = ku$, where k is a typical constant independent of u. This linear *constitutive* relationship between the elastic elongation and the applied load that causes it is the cornerstone of linear elasticity and is known as *Hook's* law.

To remove the size effect from Hook's law we replace force by *stress*, $\sigma = F/A$; by force per unit cross section area, and displacement by *strain*, $\epsilon = u/L$, to have the law written in the form $\sigma = E\epsilon$, $E = kL/A$, where E is the elastic *modulus* of the material. Capital E is the letter universally reserved for the elastic modulus. For steel, $E = 2 \cdot 10^6 Kg/cm^2$

foretelling the smallness of the elastic deformations. A steel rod of length $L = 100cm$ and cross section area $A = 1cm^2$ pulled or pushed by an end force $F = 100$kg extends or shortens by $u = 0.005cm$ only.

The general elastic problem consists of computing the displacements and internal forces in a held solid caused by a system of applied loads. In many instances movement of the elastic solid that is otherwise prevented from executing rigid body motion is so minute that the body may be considered stationary.

Returning to the mathematics of a rod under axial force F we introduce for *variable* displacement u, the force imbalance function

$$g(u) = ku - F,\ \ k > 0 \tag{9.35}$$

the roots of which are the equilibrium states of the rod. Solution of the linear *stiffness equation* $g(u) = 0$ for the one root $u' = F/k$ is simple here to the point of being unremarkable, but for more complex nonlinear elastic systems, even of one degree of freedom, solution of the single nonlinear equation can become a serious matter.

With root u' we may write the force imbalance function as $g(u) = k(u - u')$ and see that if $u < u'$, then $g(u) < 0$, and if $u > u'$, then $g(u) > 0$. The forces near $u = u'$ are *restoring* and equilibrium state $u = u'$ is *stable*. Perturbing point B from equilibrium gives rise to forces that act to restore state $u = u'$.

Virtual work $g(u)du$ can be integrated to yield *the total potential energy*

$$\pi(u) = \int_0^u g(u)du = \frac{1}{2}ku^2 - Fu \tag{9.36}$$

of the tip-loaded rod. It consists of the sum of the *elastic energy* $\mathcal{E}(u) = \frac{1}{2}ku^2$ stored in the rod extended by amount u and *potential* $\mathcal{P}(u) = -Fu$ of the applied load.

By virtue of k being strictly positive, $\mathcal{E}(u) > 0$ if $u \neq 0$, and $\mathcal{E}(u) = 0$ only when $u = 0$. At equilibrium $\pi(u') = -\mathcal{E}(u') = -\frac{1}{2}ku'^2$. Obviously $\pi'(u) = d\pi/du = g(u) = ku - F$, and the equilibrium point $u = u'$, at which $g(u') = 0$, is an *extremal point* of $\pi(u)$. Moreover, $\pi''(u) = g'(u) = \mathcal{E}''(u) = k > 0$, where prime stands for differentiation with respect to u, and the extremal point is actually a *minimum* point of $\pi(u)$. This is also readily deduced from

$$\pi(u) = \pi(u') + \frac{1}{2}k(u-u')^2 = \pi(u') + \mathcal{E}(u-u') \tag{9.37}$$

which implies that $\pi(u) \geq \pi(u')$, with equality holding for $u = u'$ only. This inequality embodies the *principle of minimum potential energy* for the tip-loaded elastic rod.

We have now two equivalent ways of stating the condition of equilibrium for the rod, one *operational* requiring that the force sum of applied force F and elastic restoring force ku at tip point B be zero, and the other *variational* ensuing from the principle of virtual work, requiring that the total potential energy of the forced elastic rod be extremal. A *minimum* of $\pi(u)$ signifies *stable* equilibrium due to restoring forces, and a *maximum* of $\pi(u)$ signifies *unstable* equilibrium due to dispersing forces.

Admittedly, it does not appear to be much of a gain to integrate $g(u)du$ and then differentiate $\pi(u)$ to produce the equation of equilibrium, but we shall see that for complex elastic systems the variational formulation holds certain surprising advantages over the operational.

Before moving on to the subject of nonlinear one degree of freedom elastic systems we briefly consider a case of displacement dependent loads appearing in the rotating rod with an end mass as shown in Fig.9.6. Here

$$\mathcal{E}(u) = \frac{1}{2}ku^2, \mathcal{P}(u) = -m\omega^2(Ru + \frac{1}{2}u^2), \pi(u) = \mathcal{E}(u) + \mathcal{P}(u) \tag{9.38}$$

and

$$\pi' = g(u) = ku - m\omega^2(R+u), \pi'' = g' = k - m\omega^2 \tag{9.39}$$

so that

$$u' = R/(\frac{k}{m\omega^2} - 1). \tag{9.40}$$

Equilibrium of the rotating rod with end mass m is stable when $k - m\omega^2 > 0$, and unstable when $k - m\omega^2 < 0$. A critical limiting angular velocity $\omega = \sqrt{k/m}$ exists for which $u' = \infty$.

9.5 Material and geometry nonlinearities

Nonlinearities complicate matters suddenly. For some materials and in the presence of very large strains Hook's linear stress-strain relationship is an oversimplification, and a nonlinear stress-strain law may be required for a better description of the rod's elastic behavior. Displacement dependent elastic coefficients such as

$$k = k(u) = k_0(1 + \alpha u^2), k_0 > 0 \tag{9.41}$$

in which k_0 and α are constants, manifest what we term material nonlinearity. A nonlinear material for which $\alpha > 0$ is said to be *hardening* with u, and this is usually the case, while a negative α means that the material *softens* with u. We shall assume throughout that $\alpha > 0$.

For the nonlinear rod

$$g(u) = k_0(u + \alpha u^3) - F,\ \mathcal{E}(u) = \frac{1}{2}k_0(u^2 + \frac{1}{2}\alpha u^4) \tag{9.42}$$

and $\pi(u) = \mathcal{E}(u) - Fu$ is *quartic* rather than *quadratic* as in the linear case. It quickly becomes apparent how computationally difficult matters get in even the most conceptually simple nonlinear problems when we set out to solve the now *cubic* stiffness equilibrium equation

$$g(u) = k_0(u + \alpha u^3) - F = 0 \tag{9.43}$$

for specified F.

The ease with which we solved the linear equation of equilibrium is gone, and the immediate price we pay for the more realistic modeling of the rod material is a decidedly more difficult and expensive solution of the equations of equilibrium. For the problem at hand matters are mitigated by the fact that for any given $F, g(u) = 0$ has only one root by virtue of the fact that $dg/du = k_0(1 + 3\alpha u^2)$ is monotonically increasing. We shall soon encounter more examples of elastic systems that have multiple states of equilibrium, some stable, some not.

At equilibrium

$$\pi^{''}(u') = \mathcal{E}^{''}(u') = k_0(1 + 3\alpha u'^2) \tag{9.44}$$

and since both $k_0 > 0$ and $\alpha > 0$, it results that $\pi^{''}(u') > 0$, implying that all equilibrium states of the nonlinear rod, whatever F, are stable. That $\pi(u)$ is minimal at $u = u'$ could also be deduced from the expression

$$\pi(u) = \pi(u') + \frac{1}{2!}(u - u')^2\pi^{''}(u') + \frac{1}{3!}(u - u')^3\pi^{'''}(u') + \frac{1}{4!}(u - u')^4\pi^{''''}(u') \tag{9.45}$$

which with

$$\pi'(u') = 0, \pi''(u') = k_0(1 + 3\alpha u'^2), \pi'''(u') = 6k_0\alpha u', \pi''''(u') = 6k_0\alpha \tag{9.46}$$

becomes

$$\pi(u) = \pi(u') + \frac{1}{2}k_0(u - u')^2(1 + \frac{1}{2}\alpha(u + u')^2 + \alpha u'^2) \tag{9.47}$$

and $\pi(u) \geq \pi(u')$, with equality holding only at $u = u'$. Near the solution $u = u'$

$$\pi(u) = \pi(u') + \frac{1}{2}k_0(1 + 3\alpha u'^2)(u - u')^2 \tag{9.48}$$

meaning that close to $u', \pi(u)$ is quadratic in $u - u'$.

A nonlinear equation, algebraic or transcendental, is best solved numerically. Solution procedures of this kind are invariably based on successive linearizations that constitute an iterative scheme to improve an initial guess for u'. A cubic equation can still be solved algebraically in terms of radicals, but the solution is cumbersome and we also have in mind the generally more involved case and, most importantly, *systems* of nonlinear equations that cannot be solved but iteratively.

The *Newton-Raphson* method for the solution of the nonlinear equation is simple, extendable to systems of nonlinear equations, and generally fast-converging. Seeking to solve nonlinear $g(u) = 0$ we start with an initial guess u_0 for which usually $g(u_0) \neq 0$, and obtain the differential correction du in $u_1 = u_0 + du$ from the linearization

$$g(u_0 + du) = g(u_0) + g'(u_0)du = g_0 + g_0'du = 0 \tag{9.49}$$

as $du = -g_0/g_0'$ or

$$u_1 = u_0 - g_0/g_0', \;\; g'(u_0) = (dg/du)_0, \tag{9.50}$$

which establishes the iterative cycle.

In our case

$$g(u) = k_0(u - u') + k_0\alpha(u^3 - u'^3), g'(u) = k_0(1 + 3\alpha u^2) \tag{9.51}$$

and

$$u_1 - u' = \alpha(u_0 - u')^2 \frac{2u_0 + u'}{1 + 3\alpha u_0^2} \tag{9.52}$$

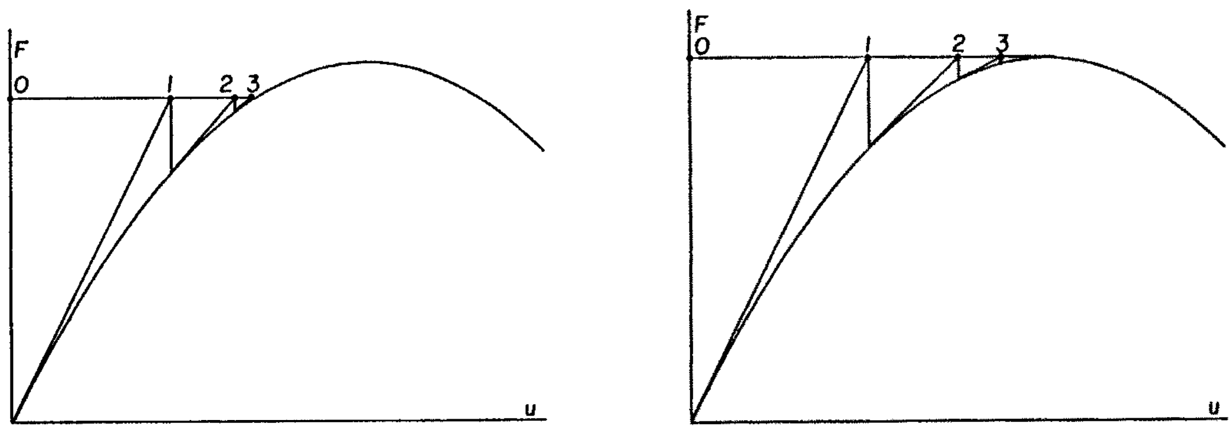
or close enough to u'

$$u_1 - u' = (u_0 - u')^2 \frac{3\alpha u'}{1 + 3\alpha u'^2} \tag{9.53}$$

which is

$$u_1 - u' = \frac{1}{2}(u_0 - u')^2 \frac{g''(u')}{g'(u')}. \tag{9.54}$$

If $g'(u') = \pi''(u') \neq 0$, then near the solution the error in the kth step of the Newton-Raphson method is nearly proportional to the *square* of the error in the $(k-1)$th step. Figure 9.7 shows the convergence progress of the Newton Raphson method applied to the solution of $g(u) = 0$ with a given F.



(a) Fig 9.7 (b)

Equilibrium at which not only $g(u') = 0$ but also $g'(u') = \pi''(u') = 0$ is said to be *indifferent*. It is an equilibrium state that is not stable but also not unstable. The convergence rate of the Newton- Raphson method to such a point, as in Fig.9.7(b), drops from quadratic to linear.

In the example of the extended rod nonlinearity is not purely material since dependence on u in $k = k_0(1 + \alpha u^2)$ is manifested only if u is large enough for αu^2 to become noticeable relative to one. If not, then the elastic constant reverts to $k = k_0$. As u diminishes the nonlinear solution approaches the linear. See Fig.9.8. We count this problem among the materially nonlinear because the configuration of the problem remains essentially unchanged under deformation. It is finite rotation that introduces the truly geometric nonlinearities where large movements compel us to compute the elastic equilibrium on the unknown *deformed* geometry, even if the elastic extensions, or strains, remain small. Such large deformation may consist of large rigid body movements with small elastic stretches superposed on them.

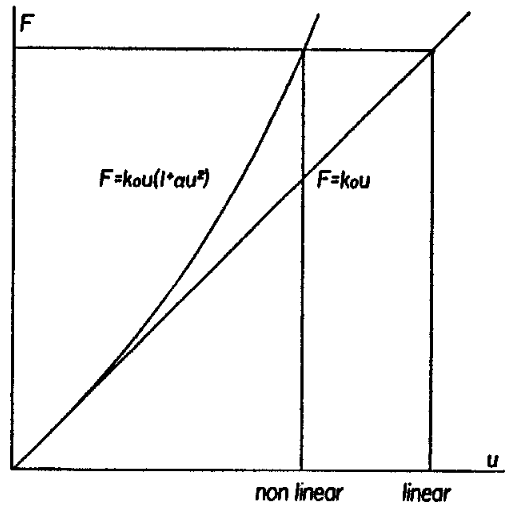


Fig. 9.8

To keep the analysis simple we discount non-axial forces that bend the rod out of its straight shape. Rotation of the rod is allowed through attachment of one rod to another or to an anchored support by means of frictionless pin joints. Such joints offer no resistance to rotation around them and cannot transmit moments. Consider Fig.9.9. Since pin joint A cannot react to a moment the forces at point B must add up to resultant F that is coaxial with rod AB if equilibrium is to be maintained.

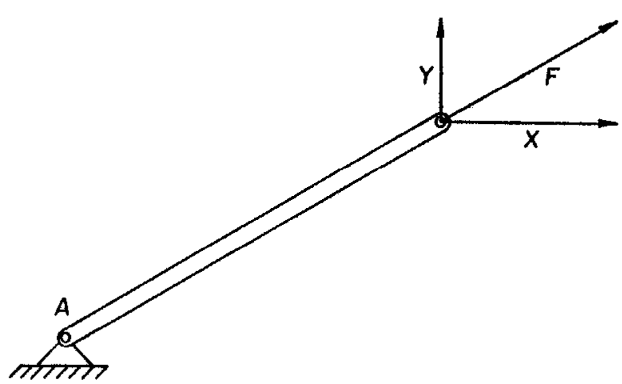


Fig. 9.9

A simple instructive example exposing some of the basic computational issues associated with geometric nonlinearity is provided by the linkage in Fig.9.10, consisting of two equal rods pin-jointed to each other at point B, and each to an immovable support at points S and S'.

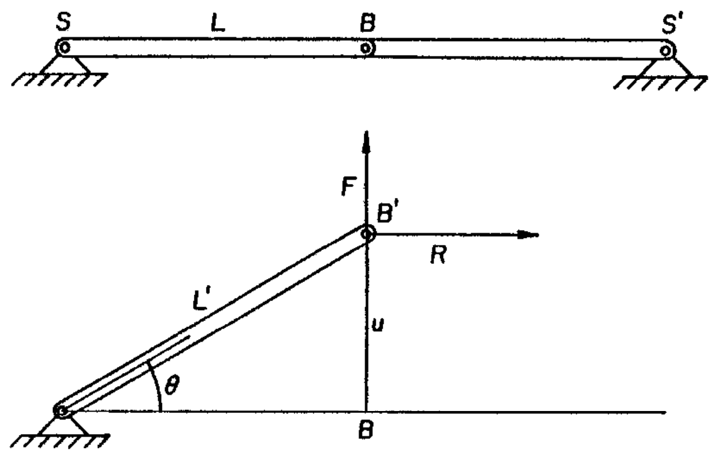


Fig. 9.10

Force $2F$ grips at point B orthogonally to axis SS'. By virtue of geometrical symmetry and the fact that the two rods are elastically identical the problem is symmetric and we may conveniently consider only one rod, say the left hand side one, allowed to rotate around point S but restricted to have its other end on perpendicular BB'. External force F pulls the rod up, while force R, the reaction of the other rod, prevents point B' from deviating sideways. The problem is thus of only one degree of freedom-vertical movement u of tip point B. We also assume that the two-rod system is under initial tension p_0 created through an initial stretch, and that the rods are both made of a linear elastic material of constant k.

Due to movement u the rod extends from length L to length L'. Calling v the stretch of the rod we obtain it in terms of u as

$$L' - L = v = \sqrt{L^2 + u^2} - L \tag{9.55}$$

with which we write the increased tension of the rod

$$p = kv + p_0. \tag{9.56}$$

The vertical component of the tension is pu/L', and at point B'

$$g(u) = \frac{pu}{L'} - F. \tag{9.57}$$

But $(u/L')du = dv$, and

$$g(u)du = pdv - Fdu \tag{9.58}$$

is an exact differential yielding, upon integration, the total potential energy

$$\pi(u) = \frac{1}{2}kv^2 + p_0 v - Fu,\ v = \sqrt{L^2 + u^2} - L \tag{9.59}$$

of the rotated rod.

We ignored reaction R in the virtual work since u is always orthogonal to it. The reaction at support S adds nothing to the virtual work since it is fixed, and the moment at the pin joint is zero.

In terms of angle θ between SB and SB'

$$g(\theta) = kL(\tan\theta - \sin\theta) + p_0 \sin\theta - F \tag{9.60}$$

and $g(\theta) = 0$ requires the solution of a transcendental equation.

In case of small displacements, that is in case $u/L << 1$, $v = \frac{1}{2}L(u/L)^2$, and the total potential energy of the rod reduces to

$$\pi(u) = \frac{1}{8}kL^2(\frac{u}{L})^4 + \frac{1}{2}p_0 L(\frac{u}{L})^2 - Fu \tag{9.61}$$

or

$$\pi(u) = \frac{1}{2}p_0 L(\frac{u}{L})^2(1 + \frac{kL}{4p_0}\frac{u^2}{L^2}) - Fu. \tag{9.62}$$

If the initial tension is positive and sufficiently large so that $ku^2/4p_0L << 1$, then the total potential energy of the rod reduces to the quadratic

$$\pi(u) = \frac{1}{2}p_0 L\frac{u^2}{L^2} - Fu \tag{9.63}$$

and the equation of equilibrium becomes linear. But if $p_0 = 0$, then

$$\pi(u) = \frac{1}{8}k\frac{u^4}{L^2} - Fu, \pi'(u) = \frac{1}{2}k\frac{u^3}{L^2} - F, \pi''(u) = \frac{3}{2}k\frac{u^2}{L^2} \tag{9.64}$$

and *the equation of equilibrium does not admit a linearization*. In the absence of an initial tension a very small F causes relatively large vertical movement of midpoint B in the sense that $ku'/F = (2k^2L^2/F^2)^{1/3} \to \infty$ as $F \to 0$.

An initial *compression* $p_0 < 0$ imposed on the two-rod system raises the possibility of three distinct equilibrium configurations even for zero F. They are the three real roots

$$u' = 0, u' = \pm\frac{p_0}{k}(1 - 2\frac{Lk}{p_0})^{1/2} \tag{9.65}$$

of equilibrium equation $pu/L' = 0$, corresponding to $u = 0$ and $p = 0$.

To decide the stability of the equilibrium we insert $u = 0$ and $p = 0$ into $\pi^{''}(u)$ and obtain

$$\pi^{''}(u = 0) = \frac{p_0}{L} < 0 \text{ and } \pi^{''}(p = 0) = \frac{ku'^2}{L^2 + u'^2} > 0 \tag{9.66}$$

leading us to the conclusion that $u' = 0$ is an *unstable* equilibrium configuration, while the other two configurations are stable. One easily imagines the compressed linkage snapping from a straight position to one of the other two stable equilibrium configurations if given a slight push. To which of the two equilibrium states the rod will jump depends on the bias in the perturbation. Notice that in the absence of initial tension, when $p_0 = 0$, $\pi^{''}(0) = 0$, and this trivial equilibrium state is *indifferent*.

Figure 9.11 shows the total potential energy of the rod for zero load, $F = 0$, as given by

$$\pi(u) = \frac{1}{2}kv^2 + p_0v - Fu, v = L(\sqrt{1 + (\frac{u}{L})^2} - 1) \tag{9.67}$$

for different values of the initial tension (compression) p_0. For $p_0 > 0$, $\pi(u)$ is essentially parabolic, for $p_0 = 0$, $\pi(u)$ is quartic near $u = 0$; and for $p_0 < 0$, $\pi(u)$ possesses a relative maximum at $u = 0$, and two minima symmetrically situated with respect to $u = 0$.

Say the rods are made such that $kL = 1$. Newton-Raphson solution of equilibrium equation $g(\theta) = \tan\theta + (p_0 - 1)\sin\theta - F = 0$ for given F confronts us with the fact that convergence of the method can occasionally be slow, erratic or even entirely absent. Figure 9.12 describes F versus θ for $p_0 = -0.5$. As θ is increased above zero a negative F is needed first to restrain the rod from being pushed out by the negative initial tension. At $\theta = 29.123^o$ the force dips maximally to $F = -0.1729$, then it decreases in magnitude to become zero at

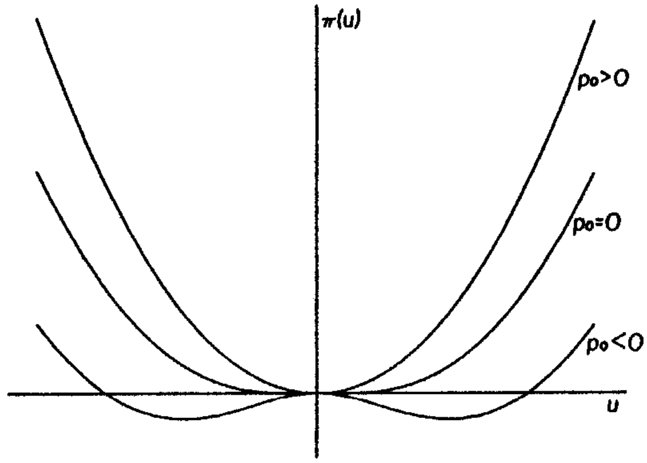


Fig. 9.11

$\theta = 48.186^o$, and keeps thereafter ever increasing with θ. Starting with $\theta_0 = 0$ and a positive F below $F = 0.173$ the Newton-Raphson method converges to a negative θ of eight digits accuracy in no more than three steps. Close to peak value $F = 0.173$ convergence slows considerably for theoretical reasons clear to us. Raising F slightly above $F = 0.173$ we lose convergence altogether, but for higher values of the applied load, say $F = 0.18$, the iterative procedure wanders around for some 16 iterations, then happens to fall upon an equilibrium point.

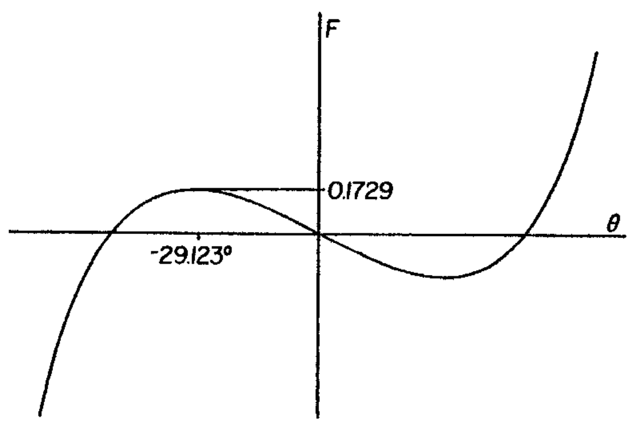


Fig. 9.12

9.6 Rod systems—the truss

Having become acquainted with the linear and nonlinear behavior of the one rod we move on to examine the highly practical question of the equilibrium and stability of elastic systems consisting of a large number of end connected rods—what we call a truss.

A truss (frame, lattice-work) is a lightweight web of rods designed to transmit structural loads to anchored supports. A two-dimensional Fink roof spacer calculated to carry the weight of a roof to the wall pillars is shown in Fig.9.13. In practice the rods are interconnected by welds, nails, bolts, or rivets but we shall follow the common practice of simplifying the model, without greatly denying reality, of assuming pin joints (ball joints in space) such as found in bicycle chains. By adding to this the restriction of external forces applied to joints only we limit the reaction forces in the rods to axial.

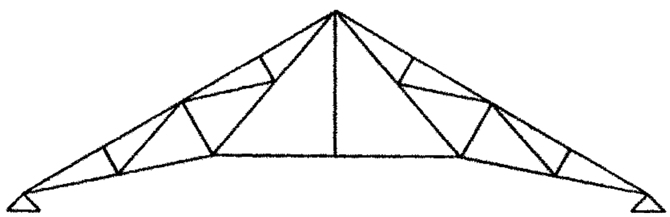


Fig. 9.13

As a result of the application of external loads, as say when a train passes over a bridge, the truss deforms slightly, changes its configuration, causing some rods to extend and some to contract giving rise thereby to tensional and compressive reactions in the rods. The purpose of our elastic equilibrium analysis is to compute the truss movement as given by the movement of its joints or nodes and the internal forces in each rod. In this, the rods are assumed utterly resistant to bending and remain straight. At first we shall assume the truss movement under the action of the external loads so small as to practically leave the truss in its original geometry.

A structural truss has to behave like a rigid body; it must not include mechanisms that allow a continuous change of configuration that leave the length of its rods unchanged. Some mechanisms allow a finite relative movement of the nodes and some only infinitesimal. The two-rod system of Fig.9.10 includes an infinitesimal mechanism, point B offering no *incipient* resistance to differential movement du at $u = 0$. The change of length $L' - L$ of rod SB resulting from small vertical movement u of point B is only *quadratic*, or second-order, in $u, (L' - L)/L = u^2/(2L^2)$, if $u << 1$. However, as u increases the two-rod system stiffens and the mechanism disappears. At a finite rise of point $B, dL' = (u/L')du$.

A mechanism-devoid truss, a truss whose configuration can be infinitesimally changed by a first-order infinitesimal lengthening or shortening of its rods only, is said to be *stiff*.

A triangular truss is stiff, while a rectangular truss is not. Not only infinitesimal, but also finite, continuous, changes of configuration exist for the rectangular truss that are accomplished without straining its four rods. A rectangular truss with one diagonal is stiff, but it can, nonetheless, have *isolated* configurations of zero strain. Forcing one triangular part into another creates another, noncongruent, quadrilateral with rods of the same length as the original. See Fig.9.14. In mechanics the violent passage of one unstrained configuration into another is called *snap-through*.

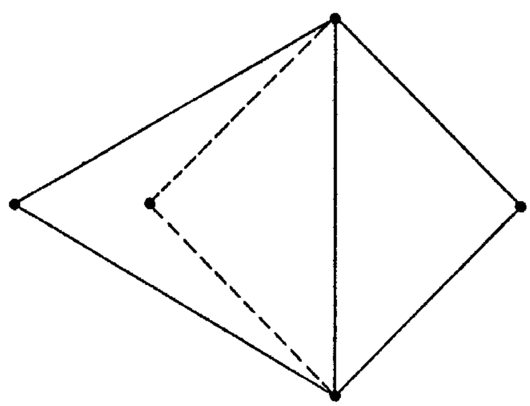


Fig 9.14

A load bearing truss needs to be stiff, but we readily imagine instances of folding frameworks with a variety of deliberate mechanisms and linkages, possibly including intended slight snap-throughs designed to click lock the structure into position.

Theorem 9.5 *The number of rods sufficient to stiffen a pin-jointed plane truss of k nodes is $2k - 3$, $k > 2$. The number of rods sufficient to stiffen a space truss is $3k - 6$.*

Proof. A two-rod system with one joint is not stiff. An additional rod completing a nontrivial triangle denies the mechanism. Each additional node calls for two connecting rods. Thus 3 nodes require 3 rods, 4 nodes require $3+2$ rods, 5 nodes require $3+2\cdot 2$ rods, 6 nodes require $3+2\cdot 2\cdot 2\cdot 2$ rods, and k nodes require $3+2(k-3)=2k-3$ rods.

In space, each additional node requires 3 connecting rods to complete a triangle to a nontrivial tetrahedron, and $3+3(k-3)=3k-6$ rods suffice to stiffen a truss of k nodes. End of proof.

The proper number of rods does not guarantee stiffness unless the connections are done purposefully. A rod taken out from a vital connection and put where it is not needed can leave part of the truss a mechanism and part overstiff. Figure 9.15 shows a truss with the right number of nodes and rods that is partly overstiff and partly understiff. Rod P_1P_2 placed at P_1P_3 restores overall stiffness. Placement of the rods is important and we shall give it a linear algebraic consideration in the next section. How to achieve certain structural objectives with the minimal number of rods arranged in some optimal configuration is a fascinating question that is too complex to be touched upon here.

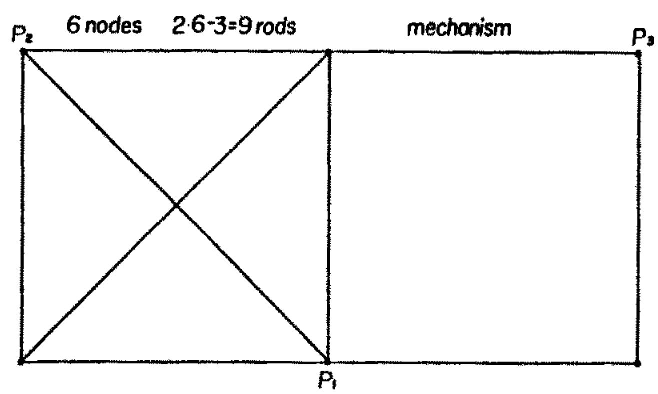


Fig 9.15

A polyhedron is a surface extending in space and consisting of polygonal faces. A tetrahedron is a polyhedron consisting of four triangular faces neatly fitting at six straight edges. A polyhedron is not a skeletal space truss consisting of bar edges, but is rather a solid. A *convex* polyhedron has only outwardly protruding vertices all of which may be touched by planes that are entirely outside the solid. All tetrahedra are convex.

Theorem (Cauchy) 9.6. *A convex polyhedron is stiff.*

We shall not attempt to prove this important theorem but remark that it refers to surfaces of solid flat faces that are by themselves absolutely rigid. we may think of the polyhedron of Cauchy's theorem as consisting of rigid laminae hinged together along their common edges. A mechanism in such a structure happens when its faces become able to swing relative to each other around the edges without change of form, the way a door swings on its hinges. A cube made of six rigid faces is, according to Caucy's theorem, rigid since no relative movement of the faces is possible that does not alter their square shape into a parallelogram. A cubical truss made of twelve bars hinged at eight vertices is, on the other hand,collapsible.

Cauchy's theorem has, nevertheless, an obvious corollary of important bearings to space frames.

Corollary 9.7. *A convex polyhedral space truss of triangular cell faces is stiff.*

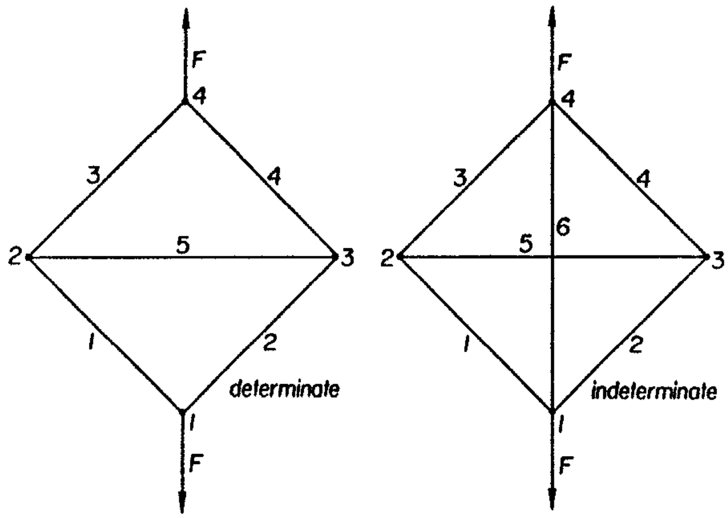
A truss with the minimum number of rods to avoid a mechanism is said to be *properly stiff*. It may well happen that for some rod configuration the internal forces in all rods can be computed from statics alone, with no recourse to elastic deformations; that is, from the equilibrium conditions for the nodes, knowing that all reaction forces are parallel to the rods. A truss for which this is possible is said to be *determinate*. A triangle in equilibrium is determinate. A truss for which the internal reaction forces in its members can be computed only with knowledge of the elastic deformation is said to be *indeterminate*.

Theorem 9.8 *A properly stiff truss is determinate.*

Proof. Removal of typical rod P_1P_2 creates a mechanism that allows joints P_1 and P_2 to come closer or move further apart. Reaction F in rod P_1P_2 is put as external forces at nodes P_1 and P_2 parallel to P_1P_2. A virtual displacement given to the mechanism provides the zero virtual work equation to determine F. End of proof.

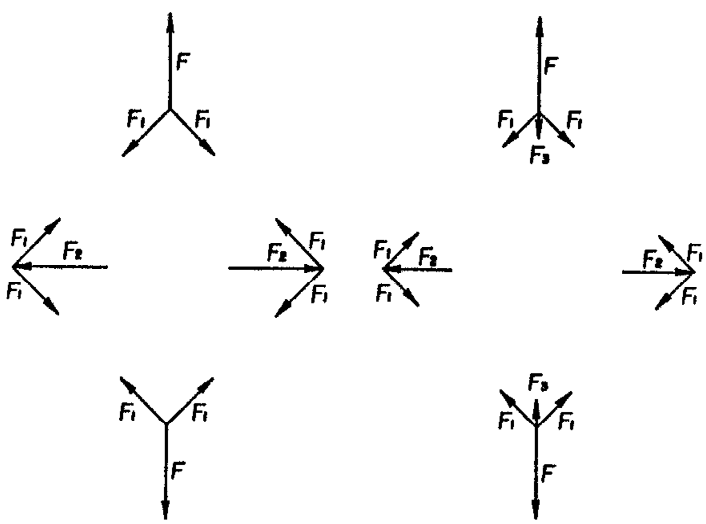
The method of finite elements to be discussed in the next section makes no distinction between a determinate and an indeterminate truss and calculates the reactions in the rods solely from elastic extensions.

Figure 9.16 shows a determinate four nodes, five rods truss, and an indeterminate four nodes, six rods truss. With geometric and elastic symmetry the external and reaction forces



(a) Fig 9.16 (b)

at the four nodes are as shown in Fig.9.17. For the truss of Fig.9.16(a) we obtain from equilibrium at point 1 (or equally well point 4) that $F_1 = \sqrt{2}/F$, and from equilibrium at point 2 (or equally well point 3) that $F_2 = F$, and all unknown internal forces are thus determined for truss 9.16(a). Truss 9.16(b) includes the extra unknown F_3, exerted on nodes 1 and 4 by rod 6, but there are only two equations of equilibrium, and the three F_1, F_2, F_3 cannot be determined.



(a) Fig 9.17 (b)

Let P_1P_2 in Fig.9.18 be a typical rod member of a two-dimensional truss restricted to movements in its plane. Deformation of the truss under the action of external loads in equilibrium causes the rod to move to position $P_1'P_2'$. Part of the motion consists of rigid body translation and rotation, and part of elastic stretch. If the original end points of the rod are $P_1(x_1, y_2), P_2(x_2, y_2)$, and the new ones $P_1'(x_1', y_1'), P_2(x_2', y_2')$, then

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \text{ and } L' = \sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2} \tag{9.68}$$

are the original and extended lengths of the rod, respectively. From the strain $\epsilon = (L' - L)/L$, and assuming linear elastic behavior, we obtain the stress $\sigma = E\epsilon$, where E is the elastic modulus of the material the rod is made of.

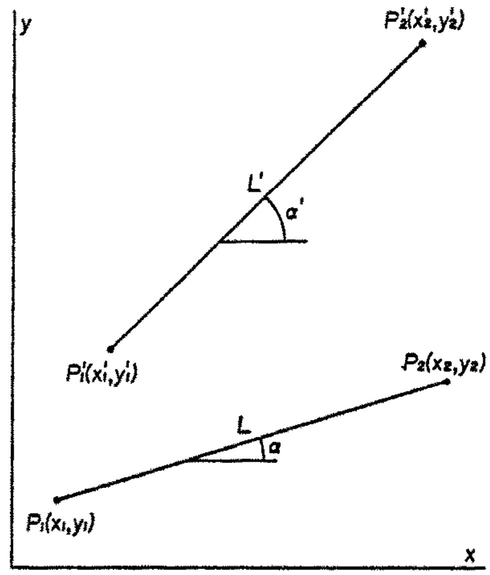


Fig 9.18

We may find it convenient to work with the relative movements of the rod's end points rather than their coordinates, and we write

$$\begin{aligned} x_1' &= x_1 + u_1 \qquad & x_2' &= x_2 + u_2 \\ y_1' &= y_1 + v_1 \qquad & y_2' &= y_2 + v_2 \end{aligned} \tag{9.69}$$

to have

$$\begin{aligned} L'^2 = L^2 + (u_2 - u_1)^2 + (v_2 - v_1)^2 &+ 2(u_2 - u_1)(x_2 - x_1) \\ &+ 2(v_2 - v_1)(y_2 - y_1). \end{aligned} \tag{9.70}$$

In case of small displacements, when $|u_2 - u_1|/L << 1$ and $|v_2 - v_1|/L << 1$, new length L' reduces to

$$L' = L\sqrt{1 + 2\frac{u_2 - u_1}{L}\frac{x_2 - x_1}{L} + 2\frac{v_2 - v_1}{L}\frac{y_2 - y_1}{L}} \tag{9.71}$$

or

$$L' = L(1 + \frac{u_2 - u_1}{L}\cos\alpha + \frac{v_2 - v_1}{L}\sin\alpha) \tag{9.72}$$

which is a linear function of the end displacements, and the small strain experienced by the rod becomes

$$\epsilon = \frac{L' - L}{L} = \frac{u_2 - u_1}{L}\cos\alpha + \frac{v_2 - v_1}{L}\sin\alpha. \tag{9.73}$$

Notice that if $\alpha = 0$, then the strain becomes due to the horizontal difference $u_2 - u_1$ only; a small vertical difference $v_2 - v_1$ in the nodes movement adds nothing to the strain of the horizontal rod.

In space, rod $P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2)$ displaces to $P_1'(x_1', y_1', z_1')$, $P_2(x_2', y_2', z_2')$ where

$$\begin{aligned} x_1' &= x_1 + u_1 \qquad & x_2' &= x_2 + u_2 \\ y_1' &= y_1 + v_1 \qquad & y_2' &= y_2 + v_2 \\ z_1' &= z_1 + w_1 \qquad & z_2' &= z_2 + w_2 \end{aligned} \tag{9.74}$$

and if $|u_2 - u_1|, |v_2 - v_1|, |w_2 - w_1|$ are very small compared with the original length L of the rod, then the extended length L' of the rod becomes a linear function of the displacements,

$$L' = L(1 + c_1\frac{u_2 - u_1}{L} + c_2\frac{v_2 - v_1}{L} + c_3\frac{w_2 - w_1}{L}) \tag{9.75}$$

where

$$c_1 = \frac{x_2 - x_1}{L}, c_2 = \frac{y_2 - y_1}{L}, c_3 = \frac{z_2 - z_1}{L} \tag{9.76}$$

are the direction cosines of the rod.

We are ready now to write down the system of equations that express the global equilibrium of the truss. Obviously, the truss is in equilibrium if and only if all its nodes are in equilibrium. Consider typical joint j of Fig.9.19 connected by three rods to neighboring joints k, l, m, and with force F_j applied to it. Relative movement of nodes j, k, l, m strains the three rods causing stresses $\sigma_1, \sigma_2, \sigma_3$ to appear in them. Reaction forces $F_1' = A_1\sigma_1, F_2' = A_2\sigma_2, F_3' = A_3\sigma_3$ in the rods are parallel to the rods and the two equations of equilibrium for node j are

$$\begin{aligned} F_1'\cos\alpha_1 + F_2'\cos\alpha_2 + F_3'\cos\alpha_3 + X_j = 0 \\ F_1'\sin\alpha_1 + F_2'\sin\alpha_2 + F_3'\sin\alpha_3 + Y_j = 0. \end{aligned} \tag{9.77}$$

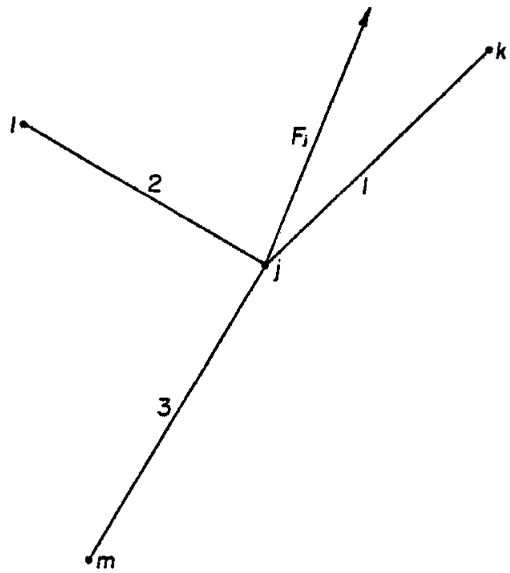


Fig 9.19

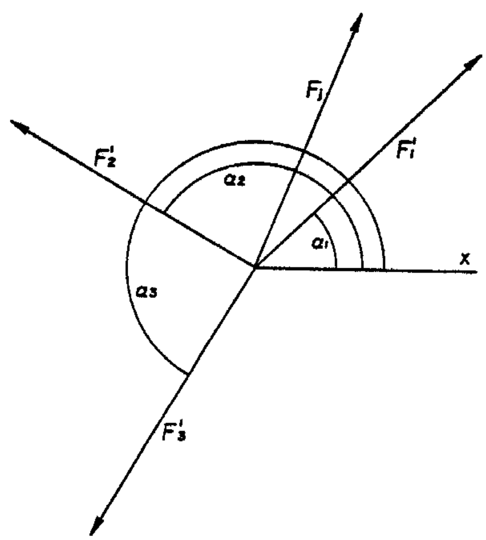


Fig 9.20

See Fig.9.20.

We recall that the strain in rod P_1P_2 due to small movement u_1, v_1 of node P_1 and small movement u_2, v_2 of node P_2 is

$$\epsilon = \frac{1}{L}\left((v_2 - v_1)\sin\alpha + (u_2 - u_1)\cos\alpha\right) \tag{9.78}$$

where L is the length of the rod and α its inclination. Assuming linear elasticity, $\sigma = E\epsilon$, the stresses in the three rods become

$$\begin{aligned}\sigma_1 &= \frac{E}{L_1}\left((u_k - u_j)\cos\alpha_1 + (v_k - v_j)\sin\alpha_1\right)\\ \sigma_2 &= \frac{E}{L_2}\left((u_l - u_j)\cos\alpha_2 + (v_l - v_j)\sin\alpha_2\right)\\ \sigma_3 &= \frac{E}{L_3}\left((u_m - u_j)\cos\alpha_3 + (v_m - v_j)\sin\alpha_3\right).\end{aligned} \tag{9.79}$$

Substitution of the three stresses into the two equations of equilibrium writes them in terms of node displacements $u_j, v_j; u_k, v_k; u_l, v_l; u_m, v_m$.

In this way $2k$ equations of equilibrium with $2k$ unknown displacements are written for the k-nodes truss. A well designed truss does not include mechanisms, but the system of equilibrium equations is still singular due to the existence of rigid body modes. Correct anchoring removes the singularity and the system is made to yield a unique displacement solution.

9.7 Linear finite elements

The elastic energy stored in an elastically linear rod of length L, cross section area A, and elastic modulus E is obtained from the integration of $d\mathcal{E} = A\sigma d(L\epsilon)$ as

$$\mathcal{E} = \mathcal{E}(x'_1, x'_2, y'_1, y'_2) = \frac{1}{2}VE\epsilon^2, V = AL \tag{9.80}$$

where $\epsilon = (L' - L)/L$ is the strain. No assumption about smallness of displacement is made in the derivation of the above expression for $\mathcal{E}$, only that the material follows the linear stress-strain constitutive relationship $\sigma = E\epsilon$.

Partial differentiation of elastic energy $\mathcal{E}$ of a typical rod yields

$$\begin{aligned}\frac{\partial\mathcal{E}}{\partial x'_1} = \frac{\partial\mathcal{E}}{\partial u_1} = VE\epsilon\frac{\partial\epsilon}{\partial x'_1} \qquad \frac{\partial\mathcal{E}}{\partial x'_2} = \frac{\partial\mathcal{E}}{\partial u_2} = VE\epsilon\frac{\partial\epsilon}{\partial x'_2}\\ \frac{\partial\mathcal{E}}{\partial y'_1} = \frac{\partial\mathcal{E}}{\partial v_1} = VE\epsilon\frac{\partial\epsilon}{\partial y'_1} \qquad \frac{\partial\mathcal{E}}{\partial y'_2} = \frac{\partial\mathcal{E}}{\partial v_2} = VE\epsilon\frac{\partial\epsilon}{\partial y'_2}.\end{aligned} \tag{9.81}$$

But

$$\begin{aligned}\frac{\partial\epsilon}{\partial x'_1} = \frac{1}{L}\frac{\partial L'}{\partial x'_1} = -\frac{x'_2 - x'_1}{LL'} \qquad \frac{\partial\epsilon}{\partial x'_2} = \frac{1}{L}\frac{\partial L'}{\partial x'_2} = \frac{x'_2 - x'_1}{LL'}\\ \frac{\partial\epsilon}{\partial y'_1} = \frac{1}{L}\frac{\partial L'}{\partial y'_1} = -\frac{y'_2 - y'_1}{LL'} \qquad \frac{\partial\epsilon}{\partial y'_2} = \frac{1}{L}\frac{\partial L'}{\partial y'_2} = \frac{y'_2 - y'_1}{LL'}\end{aligned} \tag{9.82}$$

and

$$\frac{x_2' - x_1'}{L'} = \cos\alpha', \quad \frac{y_2' - y_1'}{L'} = \sin\alpha' \tag{9.83}$$

so that finally, with $VE\epsilon = LF$, F being the axial force that stretches the rod,

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial x_1'} = \frac{\partial \mathcal{E}}{\partial u_1} = -F\cos\alpha' \quad \frac{\partial \mathcal{E}}{\partial y_1'} = \frac{\partial \mathcal{E}}{\partial v_1} = -F\sin\alpha' \\ \frac{\partial \mathcal{E}}{\partial x_2'} = \frac{\partial \mathcal{E}}{\partial u_2} = F\cos\alpha' \quad \frac{\partial \mathcal{E}}{\partial y_2'} = \frac{\partial \mathcal{E}}{\partial v_2} = F\sin\alpha'. \end{aligned} \tag{9.84}$$

The same is true for rods allowed to move in space. Elongation of a rod strains it by $\epsilon = (L' - L)/L$ and causes, assuming linear elasticity, stress $\sigma = E\epsilon$ to appear in the rod. A positive σ means that the end forces F exerted on the rod by the joints are tension causing, and a negative σ implies compression. The elastic energy stored in the rod is again $\mathcal{E} = \frac{1}{2}VE\epsilon^2$, in which V denotes the volume of the rod and E the elastic modulus. We readily verify that differentiation of the elastic energy with respect to the end displacements is here

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial x_1'} = \frac{\partial \mathcal{E}}{\partial u_1} = X_1 \quad \frac{\partial \mathcal{E}}{\partial y_1'} = \frac{\partial \mathcal{E}}{\partial v_1} = Y_1 \quad \frac{\partial \mathcal{E}}{\partial z_1'} = \frac{\partial \mathcal{E}}{\partial w_1} = Z_1 \\ \frac{\partial \mathcal{E}}{\partial x_2'} = \frac{\partial \mathcal{E}}{\partial u_2} = X_2 \quad \frac{\partial \mathcal{E}}{\partial y_2'} = \frac{\partial \mathcal{E}}{\partial v_2} = Y_2 \quad \frac{\partial \mathcal{E}}{\partial z_2'} = \frac{\partial \mathcal{E}}{\partial w_2} = Z_2 \end{aligned} \tag{9.85}$$

giving the component of end force F in the direction of the respective displacement.

This result is of sufficient importance to deserve a formal statement.

Theorem 9.9. *Let rod P_1P_2 of length L and cross section A move by finite displacements to position $P_1'P_2'$ with new length L' and cross section remaining A. The energy stored in the rod by the elastic deformation equals $\mathcal{E} = \frac{1}{2}ALE\epsilon^2$, $\epsilon = (L' - L)/L$, where E denotes the elastic modulus. Partial differentiation of $\mathcal{E}$ with respect to an end displacement produces the component of the end straining force in the direction of that displacement.*

In case of small plane displacements, when $|u_2 - u_1| << L$ and $|v_2 - v_1| << L$, angle α' is indistinguishable from α, and

$$L' = L\left(1 + \frac{u_2 - u_1}{L}\cos\alpha + \frac{v_2 - v_1}{L}\sin\alpha\right). \tag{9.86}$$

Extended length L' of the rod becomes a linear function of its end displacements, and so does strain

$$\epsilon = \frac{L' - L}{L} = \frac{1}{L}((u_2 - u_1)\cos\alpha + (v_2 - v_1)\sin\alpha) \tag{9.87}$$

experienced by the rod. Referring to the typical eth element and returning to linear algebraic notations we write the strain as

$$\epsilon = \frac{1}{L} q_e^T u_e \tag{9.88}$$

where $q_e = [-c \ \ -s \ \ c \ \ s]^T$, $c = \cos\alpha$, $s = \sin\alpha$, and where $u_e = [u_1 \ v_1 \ u_2 \ v_2]^T$ is the displacement *nodal values vector* of the eth rod. To stress the fact that we are dealing with one single typical rod we subscript $\mathcal{E}$ by e and write the elastic energy as

$$\mathcal{E}_e = \frac{1}{2}\frac{AE}{L} u_e^T q_e q_e^T u_e = \frac{1}{2} u_e^T k_e u_e \tag{9.89}$$

with

$$k_e = \frac{AE}{L} q_e q_e^T = \frac{AE}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \tag{9.90}$$

being the *element stiffness matrix* of the typical eth rod element of a plane truss.

Element stiffness matrix k_e is obviously symmetric, positive semidefinite and of rank one, with three rigid body modes

$$r_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ r_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \ r_3 = c\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \tag{9.91}$$

that are three orthogonal eigenvectors corresponding to the three zero eigenvalues of k_e—they span the nullspace of the element matrix. Vector $q_e = [-c \ \ -s \ \ c \ \ s]^T$ is the fourth eigenvector corresponding to the only nonzero eigenvalue $\lambda = 2AE/L$ of element stiffness matrix k_e. Quadratic form $u_e^T k_e u_e$ is zero for any choice of u_e that does not elongate or shorten the rod.

In space, if $|u_2 - u_1| << L$, $|v_2 - v_1| << L$, $|w_2 - w_1| << L$, then

$$L' = L(1 + c_1\frac{u_2 - u_1}{L} + c_2\frac{v_2 - v_1}{L} + c_3\frac{w_2 - w_1}{L}) \tag{9.92}$$

where

$$c_1 = \frac{x_2 - x_1}{L},\ c_2 = \frac{y_2 - y_1}{L},\ c_3 = \frac{z_2 - z_1}{L} \tag{9.93}$$

are the direction cosines of the rod. Here

$$\epsilon = \frac{L' - L}{L} = [-c_1\ \ -c_2\ \ -c_3\ c_1\ c_2\ c_3][u_1\ v_1\ w_1\ u_2\ v_2\ w_2]^T \tag{9.94}$$

or in short $\epsilon = q_e^T u_e$, and the elastic energy $\mathcal{E}_e$ of the typical rod becomes

$$\mathcal{E}_e = \frac{1}{2}\frac{AE}{L} u_e^T q_e q_e^T u_e = \frac{1}{2} u_e^T k_e u_e \tag{9.95}$$

with

$$k_e = k_e(6 \times 6) = \frac{EA}{L} q_e q_e^T,\ q_e = [-c_1\ \ -c_2\ \ -c_3\ c_1\ c_2\ c_3]^T \tag{9.96}$$

being the element stiffness matrix of the eth space rod. Element stiffness matrix k_e is once more symmetric positive semidefinite and of rank one. Every vector orthogonal to q_e represents a rigid body mode and is an eigenvector of k_e corresponding to one of its five zero eigenvalues. Vector q_e itself is the eigenvector corresponding to the sole nonzero eigenvalue of k_e. Every choice of u_e that represents a rigid body movement of the rod renders $u_e^T k_e u_e$ zero. The only choice of u_e that yields a nonzero $u_e^T k_e u_e$ is that which leads to a stretching of the rod.

Let node numbered j be the meeting point of, say, three rods of the truss, as in Fig.9.19. Elastic energies $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ of the three rods are functions of the end displacements of the three rods only. Node movements u_j and v_j appear in $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$, and in the elastic energy of no other rod. Hence

$$\frac{\partial \mathcal{E}_1}{\partial u_j} + \frac{\partial \mathcal{E}_2}{\partial u_j} + \frac{\partial \mathcal{E}_3}{\partial u_j} = \frac{\partial}{\partial u_j}(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3) = \frac{\partial}{\partial u_j}\sum_{e=1}^{n'} \mathcal{E}_j = \frac{\partial \mathcal{E}}{\partial u_j} \tag{9.97}$$

$$\frac{\partial \mathcal{E}_1}{\partial v_j} + \frac{\partial \mathcal{E}_2}{\partial v_j} + \frac{\partial \mathcal{E}_3}{\partial v_j} = \frac{\partial}{\partial v_j}(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3) = \frac{\partial}{\partial v_j}\sum_{e=1}^{n'} \mathcal{E}_j = \frac{\partial \mathcal{E}}{\partial v_j}$$

where $\mathcal{E}$ denotes the total elastic energy of the whole truss summed over all n' rods. Let X_j and Y_j be the components of force F_j externally applied at node j. The total potential energy of the k nodes, n' elements truss, is

$$\pi = \sum_{e=1}^{n'} \mathcal{E}_e - \sum_{j=1}^{k}(X_j u_j + Y_j v_j) \tag{9.98}$$

and according to Theorem 9.9 the two equations of equilibrium of joint j are

$$\frac{\partial \pi}{\partial u_j} = 0 \quad \text{and} \quad \frac{\partial \pi}{\partial v_j} = 0. \tag{9.99}$$

Generally, if we denote the *global* vector of nodal displacements by u so that $\pi = \pi(u)$, then the $n = 2k$ equations of equilibrium of the k node truss are concisely written as

$$g = grad\ \pi(u) = o. \tag{9.100}$$

This is the variational formulation of equilibrium for the truss used in the finite element method.

Clearly, the same variational formulation that holds for the plane truss holds for the space truss.

The finite element method is a procedure to write out the system of equilibrium equations for the truss through addition of all the contributions to the system by the individual rods with their two end points rather than by completing one equation at a time. The procedure is most suitable for automatic computation, and the advent of the large computer gave the finite element method its prominence, making it the standard procedure for the large scale computations of realistic structures.

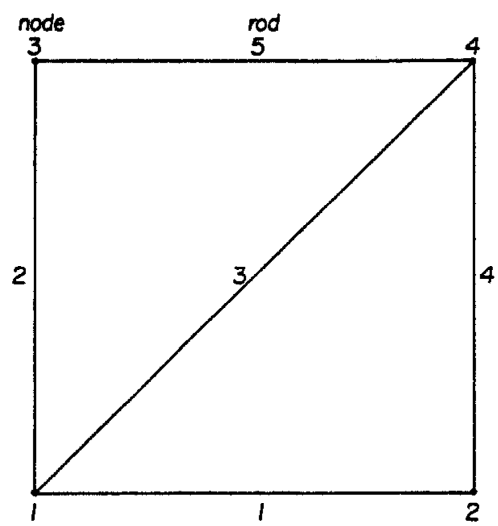


Fig 9.21

We shall give a detailed description of the finite element method on the four nodes, five rods plane truss of figure 9.21. At first we label all the nodes, here from 1 to 4. This is the global numbering system of the truss nodes. Next we label all rods, here from 1 to 5. There are two displacement unknowns at each node of the truss and hence a total of eight unknowns for the whole truss. We list them in the global displacement vector

$$u = [u_1 \; v_1 \; u_2 \; v_2 \; \dots \; u_j \; v_j \; \dots \; u_4 \; v_4]^T. \tag{9.101}$$

The two nodes of each individual rod are numbered 1 and 2, which is the element numbering system of the rod, so that always

$$u_e = [u_1 \; v_1 \; u_2 \; v_2]^T \tag{9.102}$$

whatever e. To each local node number there corresponds a global node number and we formally write this correspondence in terms of connectivity matrices A_e as

$$u_e = A_e u \quad e = 1, 2, \dots, n'. \tag{9.103}$$

For a plane truss of k nodes, $A_e = A_e(4 \times 2k)$, and in our example, where $k = 4$ and $n' = 5$

$$A_1 = \begin{array}{c} \begin{matrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \end{matrix} \\ \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \end{bmatrix} \end{array}, \quad A_2 = \begin{array}{c} \begin{matrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \end{matrix} \\ \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \end{bmatrix} \end{array}$$

$$A_3 = \begin{array}{c} \begin{matrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \end{matrix} \\ \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \end{array}, \quad A_4 = \begin{array}{c} \begin{matrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \end{matrix} \\ \begin{bmatrix} & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \end{array}$$

$$A_5 = \begin{array}{c} \begin{matrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \end{matrix} \\ \begin{bmatrix} & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \end{array}. \tag{9.104}$$

Every row of A_e contains only one single 1 and it would be inexcusably wasteful to actually store the matrix. It is a mere linear algebraic notational convention to describe the connectivity of the nodes.

To have the elastic energy of the complete structure written out in terms of the global displacement vector u we perform the transformations and summations

$$\begin{aligned}\mathcal{E} = \sum_{e=1}^{n'} \mathcal{E}_e = \frac{1}{2}\sum_{e=1}^{n'} u_e^T k_e u_e &= \frac{1}{2}\sum_{e=1}^{n'} u^T (A_e^T k_e A_e) u \\ &= \frac{1}{2} u^T \left(\sum_{e=1}^{n'} A_e^T k_e A_e \right) u \\ &= \frac{1}{2} u^T K u\end{aligned} \tag{9.105}$$

in which, for $n' = 5$

$$K = K_1 + K_2 + \ldots + K_5 \quad , \quad K_e = A_e^T k_e A_e \tag{9.106}$$

is the global stiffness matrix of the whole truss.

Transformation $K_e = A_e^T k_e A_e$ inflates k_e from a 4×4 matrix (6×6 in space) into a $2k \times 2k$ ($3k \times 3k$ in space) matrix referring to the global numbering system of the nodes. In this, entry $(k_e)_{ij}$ is sent to $(K_e)_{i'j'}$, where local i, j correspond to global i', j'. Schematically

$$K_1 = \begin{array}{c} \\ 1\\1\\2\\2\\3\\3\\4\\4 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1&1&2&2&3&3&4&4 \end{array} \\ \left[\begin{array}{cccccccc} \times&\times&\times&\times&&&& \\ \times&\times&\times&\times&&&& \\ \times&\times&\times&\times&&&& \\ \times&\times&\times&\times&&&& \\ &&&&&&& \\ &&&&&&& \\ &&&&&&& \\ &&&&&&& \end{array}\right] \end{array} , \quad K_2 = \begin{array}{c} \\ 1\\1\\2\\2\\3\\3\\4\\4 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1&1&2&2&3&3&4&4 \end{array} \\ \left[\begin{array}{cccccccc} \times&\times&&&\times&\times&& \\ \times&\times&&&\times&\times&& \\ &&&&&&& \\ &&&&&&& \\ \times&\times&&&\times&\times&& \\ \times&\times&&&\times&\times&& \\ &&&&&&& \\ &&&&&&& \end{array}\right] \end{array}$$

$$
K_3 = \begin{array}{cc} & \begin{array}{cccccccc} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \end{array} \\ \begin{array}{c} 1 \\ 1 \\ 2 \\ 2 \\ 3 \\ 3 \\ 4 \\ 4 \end{array} & \begin{bmatrix} \times & \times & & & & & \times & \times \\ \times & \times & & & & & \times & \times \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ \times & \times & & & & & \times & \times \\ \times & \times & & & & & \times & \times \end{bmatrix} \end{array}, \quad K_4 = \begin{array}{cc} & \begin{array}{cccccccc} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \end{array} \\ \begin{array}{c} 1 \\ 1 \\ 2 \\ 2 \\ 3 \\ 3 \\ 4 \\ 4 \end{array} & \begin{bmatrix} & & & & & & & \\ & & & & & & & \\ & & \times & \times & & & \times & \times \\ & & \times & \times & & & \times & \times \\ & & & & & & & \\ & & & & & & & \\ & & \times & \times & & & \times & \times \\ & & \times & \times & & & \times & \times \end{bmatrix} \end{array}
$$

$$
K_5 = \begin{array}{cc} & \begin{array}{cccccccc} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \end{array} \\ \begin{array}{c} 1 \\ 1 \\ 2 \\ 2 \\ 3 \\ 3 \\ 4 \\ 4 \end{array} & \begin{bmatrix} & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & \times & \times & \times & \times \\ & & & & \times & \times & \times & \times \\ & & & & \times & \times & \times & \times \\ & & & & \times & \times & \times & \times \end{bmatrix} \end{array} \tag{9.107}
$$

and

$$
K = K_1 + K_2 + K_3 + K_4 + K_5 = \begin{array}{cc} & \begin{array}{cccccccc} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \end{array} \\ \begin{array}{c} 1 \\ 1 \\ 2 \\ 2 \\ 3 \\ 3 \\ 4 \\ 4 \end{array} & \begin{bmatrix} \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & & & \times & \times \\ \times & \times & \times & \times & & & \times & \times \\ \times & \times & & & \times & \times & \times & \times \\ \times & \times & & & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times \end{bmatrix} \end{array} \tag{9.108}
$$

with the blanks indicating that nodes 2 and 3 are not connected. The larger the truss the larger K, and it is sparse, but not of a strict band form. If node j is connected by rods to, say, nodes k, l, m, then the two (three in space) equations of equilibrium for node j include the unknown displacements at nodes j, k, l, m only.

Global stiffness matrix K is symmetric and at least positive semidefinite. If the truss does not include a mechanism, then matrix K is of rank $2k-3$ in the plane and rank $3k-6$ in space. Every rigid body movement of the truss is an eigenvector of K corresponding to a zero eigenvalue of K. Anchoring the truss to avoid rigid body movements renders K positive definite.

We collect the external loads at the joints in vector

$$f = [X_1\ Y_1\ X_2\ Y_2\ \ldots\ X_j\ Y_j\ \ldots\ X_n\ Y_n]^T \tag{9.109}$$

and write the total potential energy of the truss as

$$\pi(u) = \frac{1}{2}u^T K u - u^T f. \tag{9.110}$$

Now grad $\pi(u) = o$ becomes

$$Ku = f \tag{9.111}$$

which is the global system of equilibrium equations.

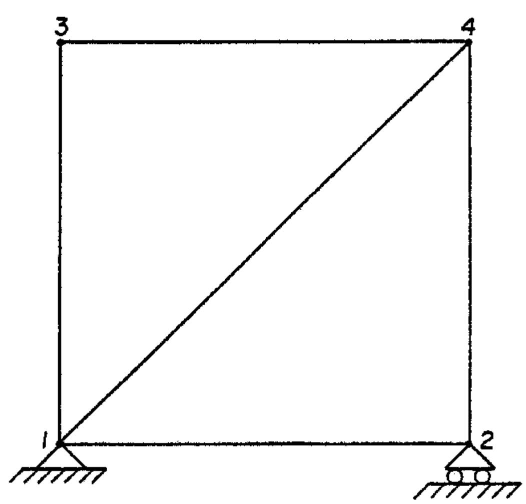


Fig. 9.22

How to incorporate into the system of equilibrium equations restrictions on joint movements is considered next. Say that our truss is tied to its abutments in the manner shown in Fig.9.22. Node 1 is fixed so that $u_1 = v_1 = 0$, and node 2 is permitted to slide horizontally only, so that $v_2 = 0$. This excludes any rigid body motion of the truss. Since u_1, v_1 and v_2 are fixed, $\partial\pi/\partial u_1 = \partial\pi/\partial v_1 = \partial\pi/\partial v_2 = 0$, and equations 1,2 and 4 of system $Ku = f$ are deleted. Substitution of $u_1 = v_1 = v_2 = 0$ into the remaining equations amounts effectively to deletion of columns 1,2 and 4 of stiffness matrix K as well. The smaller system left is positive definite and is solved for the unique elastic displacement of the truss.

Computationally it is often more convenient to leave the system in its original size and introduce the restrictions on the displacements through changes in the coefficients and right-hand side of $Ku = f$. Let's be more general and assume that some displacements have prescribed, zero or nonzero, values. For the purpose of simple discussion we assume that global displacement vector u is partitioned as $u = [u_1^T \ u_2^T]^T$ with subvector u_1 containing the prescribed values. Partitioned $Ku = f$ now has the form

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{9.112}$$

and after deletion,

$$\begin{bmatrix} I & O \\ K_{12}^T & K_{22} \end{bmatrix} u = \begin{bmatrix} u_1 \\ f_2 \end{bmatrix}. \tag{9.113}$$

Solution of the corrected system reproduces the prescribed values within u. Otherwise the system is compressed into

$$K_{22}u_2 = f_2 - K_{12}^T u_1 \tag{9.114}$$

with a positive definite and symmetric K_{22}, and is solved for unknown vector u_2.

Oblique sliding node conditions are accounted for by describing the node displacements in terms of the tangential and normal displacements u' and v', respectively, and setting $v' = 0$.

Assume that system $Ku = f$ is for a fixed truss so that K is positive definite and symmetric. To distinguish between *variable* u in $\pi(u) = \frac{1}{2}u^T Ku - u^T f$ and u at equilibrium we shall write the latter as $u' = K^{-1}f$. We readily verify that $\pi(u') = -\frac{1}{2}u'^T Ku'$. Because

K is positive definite

$$\mathcal{E}(u-u') = \frac{1}{2}(u-u')^T K (u-u') \geq 0 \tag{9.115}$$

with equality holding only when $u = u'$. It results from this that

$$\mathcal{E}(u-u') = \pi(u) - \pi(u') \geq 0 \tag{9.116}$$

and hence that

$$\pi(u') \geq \pi(u) \tag{9.117}$$

with equality holding only when $u = u'$. This is the principle of minimum potential energy for the complete structure.

Theorem 9.10 *A stiff, anchored truss that does not include a mechanism is always in stable equilibrium.*

Proof. Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the positive eigenvalues of positive definite stiffness matrix K, with corresponding eigenvectors $x_1, x_2, \ldots, x_n$. Let u be an arbitrary displacement given to the truss nodes. According to Theorem 9.9 Ku is the vector of forces that the nodes exert on the rods as a result of the elastic deformation. Since $Kx_j = \lambda_j x_j$, and since $\lambda_j > 0$, the forces on the rods due to $u = x_j$ are always in the direction of the displacements. The reaction in the rods is therefore always restoring, tending to bring the truss back to its original position. Every arbitrary u consists of a linear combination of the eigenvectors and the forces are restoring for any u. End of proof.

Iterative methods for the solution of the global stiffness equation $Ku = f$ set up with finite elements are most attractive here by virtue of the fact that such methods do not require K in tabular form but only the repeated vector matrix product Ku for variable u. This makes such iterative methods independent of the sparseness pattern of K, avoids the complexity of sparse algorithms, and leads to considerable storage savings. Indeed, we may write

$$Ku = \sum_{e=1}^{n'} (A_e^T k_e A_e) u = \sum_{e=1}^{n'} A_e^T (k_e u_e) \tag{9.118}$$

perform $k_e u_e$, expand it to the global level, and sum the contribution of each rod over the n' elements of the structure. All we need to do this is to store for each rod element its three direction cosines, its stiffness coefficient, and two integers for its global node numbers.

9.8 Suppression of rigid body movements

Imagine a free truss in space acted upon by a system of external forces in equilibrium. Linear system $Ku = f$ has a singular positive semidefinite stiffness matrix, but the system is consistent. For the sake of simplicity assume that stiffness matrix K possesses only three zero eigenvalues, with three corresponding orthogonal rigid body modes r_1, r_2, r_3. System $Ku = f$ describing the equilibrium of the truss is soluble only if the load vector f is orthogonal to the three rigid body modes, if $f^T r_1 = 0$, $f^T r_2 = 0$, $f^T r_3 = 0$,that is if f is orthogonal to the nullspace of K, implying that the system of applied loads is in equilibrium.

Displacement vector u of the truss may be decomposed into

$$u = u' + \alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3 \tag{9.119}$$

where $\alpha_1, \alpha_2, \alpha_3$ are arbitrary, and where vector u' is unique if orthogonal to r_1, r_2, r_3. We may obtain displacement vector u from the solution of singular system $Ku = f$, but handling a system of equations with a nonunique solution is cumbersome and we wish to avoid this difficulty by restraining the truss so as to compel the linear system to yield either u' or any other unique u for some $\alpha_1, \alpha_2, \alpha_3$.

We may proceed in various ways to include the constraint through modification of K. The first possibility consists of prescribing some node displacements in order to fix $\alpha_1, \alpha_2, \alpha_3$, but we need to be careful about the choice of the nodal constraints. Consider for example a free rod in the plane with $u = [u_1\ v_1\ u_2\ v_2]^T$ and $u' = [u'_1\ v'_1\ u'_2\ v'_2]$ so that

$$\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} u'_1 \\ v'_1 \\ u'_2 \\ v'_2 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} s \\ -c \\ -s \\ c \end{bmatrix}, s = \sin\alpha, c = \cos\alpha \tag{9.120}$$

for arbitrary $\alpha_1, \alpha_2, \alpha_3$. Is it possible to fix $\alpha_1, \alpha_2, \alpha_3$ by setting $u_1 = v_1 = u_2 = 0$ to have $Ku = f$ yield a unique solution? Setting in the above equation $u_1 = v_1 = u_2 = 0$ we obtain

$$\begin{aligned} \alpha_1 &= -\frac{1}{2}(u'_1 + u'_2) \\ \alpha_3 &= -\frac{1}{2s}(u'_1 - u'_2) \\ \alpha_2 &= -v'_1 + \alpha_3 c \end{aligned} \tag{9.121}$$

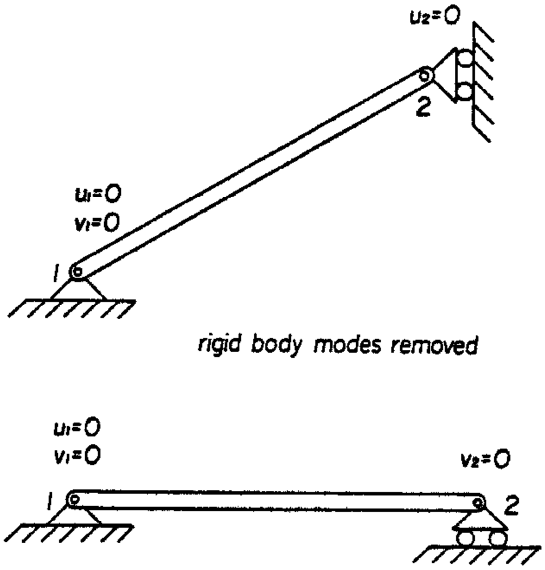


Fig. 9.23

and the answer is yes provided that $s \neq 0$. In case of a horizontal rod, when $s = 0$ and $c = 1$, we choose instead $u_1 = v_1 = v_2 = 0$ to have $\alpha_1 = -u_1', \alpha_2 = -(v_1' + v_2')/2, \alpha_3 = (v_1' - v_2')/2$. See Fig.9.23.

The second possibility consists of considering the constrained minimization of the total potential energy

$$\pi(u) = \frac{1}{2}u^T K u - u^T f,\ u^T r_1 = u^T r_2 = u^T r_3 = 0 \tag{9.122}$$

with *Lagrange multipliers*. Writing

$$\pi'(u) = \frac{1}{2}u^T K u - u^T f - \lambda_1 u^T r_1 - \lambda_2 u^T r_2 - \lambda_3 u^T r_3 \tag{9.123}$$

we obtain from grad $\pi = o$ and $\partial\pi'/\partial\lambda_1 = \partial\pi'/\partial\lambda_2 = \partial\pi'/\partial\lambda_3 = 0$ the system

$$Ku - f - \lambda_1 r_1 - \lambda_2 r_2 - \lambda_3 r_3 = 0,\ r_1^T u = r_2^T u = r_3^T u = 0. \tag{9.124}$$

Premultiplication of the first of the above equations successively by r_1^T, r_2^T, r_3^T yields $\lambda_1 = \lambda_2 = \lambda_3 = 0$, and the three constraints are added as

$$\begin{bmatrix} K \\ r_1^T \\ r_2^T \\ r_3^T \end{bmatrix} \begin{bmatrix} u \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{9.125}$$

increasing the linear system by three redundant equations.

The third possibility consists of replacing matrix K by

$$K' = K + \lambda_1 r_1 r_1^T + \lambda_2 r_2 r_2^T + \lambda_3 r_3 r_3^T \tag{9.126}$$

for any $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0$. Matrix K' becomes thereby positive definite, with eigenvalues equal to those of K except for the three zero eigenvalues that become $\lambda_1, \lambda_2, \lambda_3$. Premultiplying system $K'u = f$,with K' in eq.(9.126), by r_1^T, r_2^T, r_3^T, while recalling that $r_j^T K u = u^T K r_j = 0$ for $j = 1, 2, 3$, we verify that unique solution u of $K'u = f$ is such that $u^T r_1 = u^T r_2 = u^T r_3 = 0$.

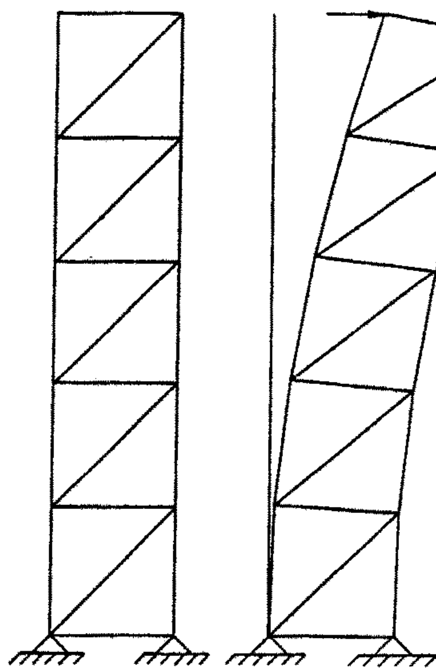
Matrix $\lambda_1 r_1 r_1^T + \lambda_2 r_2 r_2^T + \lambda_3 r_3 r_3^T$ spoils the sparseness of K', but if the method of conjugate gradients is used to solve system $K'u = f$, then for any u, $K'u = Ku + \lambda_1 (r_1^T u) r_1 + \lambda_2 (r_2^T u) r_2 + \lambda_3 (r_3^T u) r_3$, and at each iteration we merely need add to original Ku the three vectors that are proportional to r_1, r_2, r_3.

One of the first things done with global stiffness matrix K for a designed frame is the computation of

$$\min_u \mathcal{E}(u) = \min_u \frac{1}{2} u^T K u, \quad u^T u = 1 \tag{9.127}$$

which is equivalent to the computation of the lowest eigenvalue of K. A zero eigenvalue for an otherwise anchored truss means the existence of a hidden mechanism. A small first eigenvalue means that the structure includes a near-latent mechanism and a corresponding normalized deformation mode that stores little energy. If stiffness is important various truss configurations are evaluated and the one with the highest first eigenvalue is selected.

Figure 9.24 shows a tall truss as a naive representation of a high rise building. As the structure increases in height it becomes easier to swing it with the expenditure of little energy. High structures tend to become unstable, which imposes practical limitations on their height-to-width ratio. Winds and minor earthquakes may cause uncomfortable, if not dangerous, movements of the top stories of very high buildings. For slender structures of such high flexibility, their elastic instability implies very high ratios between the highest and lowest eigenvalues of stiffness matrix K, implying, in turn, the numerical instability of the equilibrium system of equations. A structure that is elastically unstable is also numerically unstable.



(a) Fig 9.24 (b)

exercises

9.8.1. Write the finite element program to assemble global stiffness matrix K for the truss tower of Fig.9.24. Use the Rayleigh quotient minimization (maximization) algorithm of Sec. 8.7 to compute the extremal eigenvalues of K. Evaluate the spectral condition number of K as it varies with height.

9.8.2. Write an efficient conjugate gradient program, as described in Sec. 7.3, to solve the finite element stiffness equation for the tower in Fig.9.24(b). Study the convergence behavior of the iterative solution for a tower of increasing height.

9.8.3. Write a Gauss band solver for the stiffness equation of the truss tower in Fig.9.24.

9.9 Nonlinear finite elements

We shall have to deal here with nonquadratic elastic energies and their differentiation. Let $\phi = \phi(\xi)$ be a scalar function of quadratic form $\xi = x^T Ax$ for variable vector x. To be specific we limit discussion to $A = A(2 \times 2)$ for which

$$\xi = A_{11}x_1^2 + 2A_{12}x_1x_2 + A_{22}x_2^2 \tag{9.128}$$

and have upon differentiation that

$$\frac{\partial \phi}{\partial x_1} = 2\dot{\phi}(A_{11}x_1 + A_{12}x_2)$$

$$\frac{\partial \phi}{\partial x_2} = 2\dot{\phi}(A_{12}x_1 + A_{22}x_2) \tag{9.129}$$

or in short

$$\frac{\partial \phi}{\partial x} = 2\dot{\phi}Ax \tag{9.130}$$

where $\dot{\phi} = d\phi/d\xi$. Derivative $\partial\phi/\partial x$ of *scalar* function ϕ with respect to variable *vector* x is a vector. Further differentiation yields

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} = 2\dot{\phi}A_{ij} + 4\ddot{\phi}r_i^T r_j \tag{9.131}$$

where $r_i = A_{i1}x_1 + A_{i2}x_2$, or in short

$$\frac{\partial^2 \phi}{\partial x^2} = 2\dot{\phi}A + 4\ddot{\phi}(Ax)(Ax)^T. \tag{9.132}$$

Second-order differentiation of scalar function ϕ with respect to variable vector x produces a symmetric matrix that is possibly a function of x.

We persist in assuming linear elasticity so that the elastic energy of the rod remains

$$\mathcal{E}_e = \frac{1}{2}AEL\epsilon^2, \quad \epsilon = (L' - L)/L \tag{9.133}$$

even under large displacements. A typical rod with end points $P_1(x_1, y_1), P_2(x_2, y_2)$ moves to position $P_1'(x_1', y_1'), P_2'(x_2', y_2')$ and we write $x_e = [x_1' \; y_1' \; x_2' \; y_2']$ since we want to describe the deformed state of the rod not by the displacements of its nodes but rather by their new positions. Element nodal vector x_e allows us to write the new length of the rod as

$$L' = \sqrt{x_e^T N x_e}, \quad N = \begin{bmatrix} 1 & & -1 & \\ & 1 & & -1 \\ -1 & & 1 & \\ & -1 & & 1 \end{bmatrix} \tag{9.134}$$

and the total potential energy

$$\pi(x) = \mathcal{E}(x) - \mathcal{P}(x) \tag{9.135}$$

of the complete truss becomes a function of vector x that contains the displaced coordinates of all nodes. Potential $\mathcal{P}(x)$ of the applied forces is usually a linear function of $x, \mathcal{P}(x) = x^T f$, and

$$g(x) = grad\ \pi = \frac{\partial \mathcal{E}}{\partial x} - \frac{\partial \mathcal{P}}{\partial x} = \frac{\partial \mathcal{E}}{\partial x} - f = o \tag{9.136}$$

is the system of equilibrium equations for the largely deformed truss. It is a nonlinear system and needs be solved iteratively.

Let x_0 be an initial guess for the equilibrium position of the truss. Usually $g(x_0) \neq o$ and following the Newton-Raphson method we seek a differential correction dx to x_0 from the linearization

$$g(x_0 + dx) = g_0 + (\frac{\partial g}{\partial x})_0 dx = o \tag{9.137}$$

in which $g_0 = g(x_0)$. Expression $(\partial g/\partial x)_0$, the derivative of vector g with respect to vector x at $x = x_0$, is a square symmetric matrix depending on x_0, that we call K_0, so that with $dx = x_1 - x_1$

$$g_0 + K_0(x_1 - x_0) = o \tag{9.138}$$

and the Newton-Raphson method for the solution of the nonlinear system $\partial \pi/\partial x = o$ is described by the iterative process

$$x_1 = x_0 - K_0^{-1} g_0. \tag{9.139}$$

We shall next describe how to carry it out with finite elements.

With the assumption that $\partial \mathcal{P}/\partial x$ is constant vector f and with the notation $g'_e = \partial \mathcal{E}_e/\partial x_e$ we have that

$$g' = \frac{\partial \mathcal{E}}{\partial x} = \sum_{e=1}^{n'} \frac{\partial \mathcal{E}_e}{\partial x} = \sum_{e=1}^{n'} A_e^T \frac{\partial \mathcal{E}_e}{\partial x_e} = \sum_{e=1}^{n'} A_e^T g'_e \tag{9.140}$$

with

$$g'_e = \frac{\partial \mathcal{E}_e}{\partial x_e} = AEL\epsilon \frac{\partial \epsilon}{\partial x_e} = \frac{AE}{L'} \epsilon N x_e \tag{9.141}$$

where, we recall, $\epsilon = L'/L - 1,\ L' = \sqrt{x_e^T N x_e}$. Now

$$g = \frac{\partial \pi}{\partial x} = g' - f = \sum_{e=1}^{n'} A_e^T g'_e - f. \tag{9.142}$$

Gradient vector $g_0 = g(x_0) = (2\pi/\partial x)_0$ is computed thus: Vector x_0 is guessed, and for the rods, one after another, element vector x_e is picked out for them from the global according to the global node numbers of the element. Quadratic form $x_e^T N x_e$ gives L', with which $\epsilon = L'/L - 1$ and $g'_e = (AE/L)\epsilon N x_e$ are computed. Expansion $A_e^T g'_e$ and summation over all n' rod elements of the truss yields g' and then gradient $g(x_0)$.

Computation of global stiffness matrix $K_0 = K(x_0)$ is similar,

$$\frac{\partial g}{\partial x} = \sum_{e=1}^{n'} \frac{\partial g'_e}{\partial x} = \sum_{e=1}^{n'} \frac{\partial^2 \mathcal{E}_e}{\partial x^2} = \sum_{e=1}^{n'} A_e^T \frac{\partial^2 \mathcal{E}_e}{\partial x_e^2} A_e \tag{9.143}$$

and

$$k_e = \frac{\partial^2 \mathcal{E}_e}{\partial x_e^2} = AEL \left(\epsilon \frac{\partial^2 \epsilon}{\partial x_e^2} + (\frac{\partial \epsilon}{\partial x_e})(\frac{\partial \epsilon}{\partial x_e})^T \right) \tag{9.144}$$

is the nonlinear element stiffness matrix of the typical eth element. But

$$\frac{\partial \epsilon}{\partial x_e} = \frac{1}{L}\frac{\partial L'}{\partial x_e} = \frac{1}{LL'} N x_e \tag{9.145}$$

and

$$\frac{\partial^2 \epsilon}{\partial x_e^2} = \frac{1}{LL'}(N - \frac{1}{L'^2}(Nx_e)(Nx_e)^T) \tag{9.146}$$

so that finally

$$k_e = \frac{AE}{L'}(\epsilon N + \frac{1}{L'^2}(Nx_e)(Nx_e)^T) \tag{9.147}$$

where A, E, L' and ϵ are all for that element. Element vector x_e and extended length L' are available from the computation of g_e, and matrix $k_e = k_e(4 \times 4)$ is readily set up. Summation of $A_e^T k_e A_e$ over all n' rods produces the here x-dependent global stiffness matrix K_0. Vector g_0 and matrix K_0 are entered into the Newton-Raphson algorithm and a better approximation x_1 is computed. If $\|g_1\|$ is sufficiently reduced the iterative procedure is stopped, if not, it is continued to hopeful convergence.

After some m steps practically $g_m = o$, and x_m holds the equilibrium configuration of the greatly deformed truss. At this point the lowest eigenvalue of $K_m = K(x_m)$ is computed to determine the stability of the equilibrium. A negative eigenvalue indicates, according to the reasoning of Theorem 9.10, that the equilibrium state of the truss is unstable and that a perturbation mode exists for which some of the created reactions are dispersing rather than restoring.

One of the basic aspects of a nonlinear structural investigation consists of tracing the deformation as a function of the load intensity. We shall assume that the acting forces are all proportional to one scalar parameter λ so that lack of load is implied by $\lambda = 0$. Before we undertake discussion of the displacement-against-force tracing for structures of many degrees of freedoms we shall look first at the more easily described case of a one degree of freedom structure whose single equation of equilibrium we write as $g(x, \lambda) = 0$.

Figure 9.25 shows a section of the equilibrium curve. To trace it we shall need points on the curve close enough for a plotter to join for a good curvilinear representation.

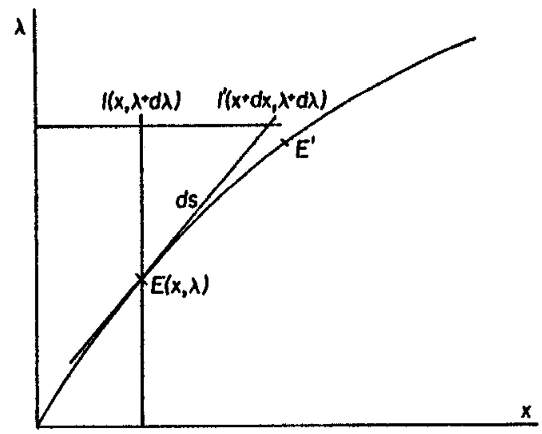


Fig 9.25

Suppose that point $E(x, \lambda)$ is found to be on equilibrium curve $g(x, \lambda) = 0$. We want now to move ahead and select a new initial guess to restart the Newton-Raphson iterative procedure to eventually land us on the next equilibrium point E'. One possibility is to start from point I of Fig.9.25, with the same x as at point E but with λ increased to $\lambda + d\lambda$. Another, more reasonable possibility consists of moving distance ds along the tangent line to $g(x, \lambda) = 0$ at point E to reach point I' of Fig.9.25. Fixing point I' is the *predictor* stage of the nonlinear solution procedure. Admittedly, the one degree of freedom problem may be too simple for such involved prediction and solution mechanism since we may vary x and simply solve $g(x, \lambda) = 0$ for λ; but we have the larger case in mind.

Since at point E, $g(x, \lambda) = 0$, linearization of $g(x + dx, \lambda + d\lambda) = 0$ yields the tangent line equation

$$g' dx + \dot{g} d\lambda = 0 \tag{9.148}$$

where $g' = dg/dx$, $\dot{g} = dg/d\lambda$, and where the derivatives are evaluated at point E. Addition of the condition

$$dx^2 + d\lambda^2 = ds^2 \tag{9.149}$$

leads to

$$d\lambda = \pm \left(\frac{g'^2}{g'^2 + \dot{g}^2} \right)^{1/2} ds, \;\; dx = \pm \left(\frac{\dot{g}^2}{g'^2 + \dot{g}^2} \right)^{1/2} ds \tag{9.150}$$

with sign choice that correspond to movement up or down the tangent line.

The next solution stage is that of *correction*, consisting of repeated solutions of the linearized equation of equilibrium. We are at initial point $I(x_0, \lambda_0)$ at which $g(x_0, \lambda_0) \neq 0$, and seek to reach some point E' on the equilibrium curve near previous point E but slightly away from it. If we prefer to carry out the Newton-Raphson iterations under constant load, then we write $g(x_0 + dx, \lambda_0) = 0$, linearize the equation as $g(x_0, \lambda_0) + g'(x_0, \lambda_0)dx = 0$, solve it for dx, correct x_0 to $x_1 + dx$, and do it over. Point N' of Fig.9.26 is where this first step of the Newton-Raphson method lands us, and if the equilibrium curve were linear, point N' would be on it. But $g = 0$ is not linear and iterations continues with the same λ_0.

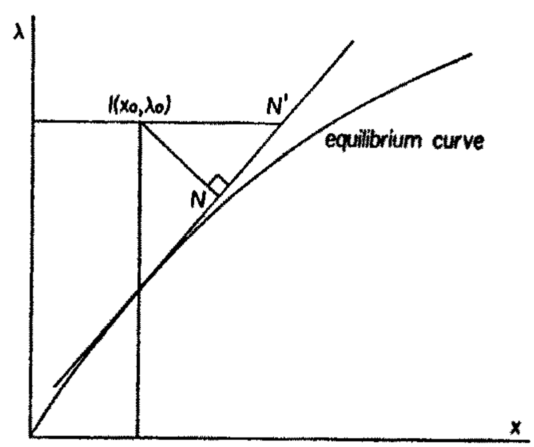


Fig 9.26

As we have seen in Section 9.5 and as is obvious from $g_0 + g_0'dx = 0$, near a critical point at which $g' = 0$, application of the Newton-Raphson method under constant load becomes difficult and we may want to vary not only x but also λ. Linearization of $g(x_0+dx, \lambda_0+d\lambda) = 0$ becomes

$$g_0 + g_0'dx + \dot{g}_0 d\lambda = 0 \tag{9.151}$$

or

$$dx = -\frac{g_0}{g_0'} - \frac{\dot{g}_0}{g_0'} d\lambda \tag{9.152}$$

and we choose $d\lambda$ by the condition that

$$dx^2 + d\lambda^2 = (\frac{g_0}{g_0'})^2 + 2(\frac{g_0 \dot{g}_0}{g_0'^2}) d\lambda + (\frac{\dot{g}_0}{g_0'})^2 d\lambda^2 + d\lambda^2 \tag{9.153}$$

be minimal. This readily leads to

$$d\lambda = -\frac{g_0 \dot{g}_0}{\dot{g}_0^2 + g_0'^2}, \ dx = -\frac{g_0 g_0'}{\dot{g}_0^2 + g_0'^2} \tag{9.154}$$

and corrected point N in Fig.9.26.

The general case where x is a vector and $g(x, \lambda) = o$ stands for a system of nonlinear equations is similar. To predict a starting point we linearize $g(x + dx, \lambda + d\lambda) = o$ as

$$(\frac{\partial g}{\partial x}) dx + (\frac{\partial g}{\partial \lambda}) d\lambda = o \tag{9.155}$$

where $K = \partial g / \partial x$ is the global stiffness matrix of the structure, and where $\partial g / \partial \lambda = f$ is the vector of applied loads. Solution of the linearized equation for dx yields

$$dx = -d\lambda K^{-1} f = -d\lambda q, \ q = K^{-1} f. \tag{9.156}$$

Restriction

$$dx^T dx + d\lambda^2 = d\lambda^2 (q^T q + 1) = ds^2 \tag{9.157}$$

yields

$$d\lambda = \frac{ds}{\sqrt{q^T q + 1}}, \quad dx = -d\lambda q \tag{9.158}$$

and prediction is done.

Linearized correction under changing load is obtained from $g(x_0 + dx, \lambda_0 + d\lambda) = o$ as

$$g_0 + K_0 dx + d\lambda f = o \tag{9.159}$$

so that

$$dx = -K_0^{-1} g_0 - d\lambda K_0^{-1} f = -p_0 - d\lambda q_0 \tag{9.160}$$

where $p_0 = K_0^{-1} g_0$ and where $q_0 = K_0^{-1} f$. Here

$$dx^T dx + d\lambda^2 = p_0^T p_0 + 2p_0 q_0^T d\lambda + (q_0^T q_0 + 1) d\lambda^2 \tag{9.161}$$

and the condition that $dx^T dx + d\lambda^2$ be minimum with respect to $d\lambda$ yields

$$d\lambda = -\frac{p_0^T q_0}{1 + q_0^T q_0}, \ dx = -p_0 - d\lambda q_0. \tag{9.162}$$

Initial guess x_0, λ_0 is corrected to $x_1 = x_0 + dx, \lambda_1 = \lambda_0 + d\lambda$ and the correction procedure is repeated until $\|g(x_m, \lambda_m)\|$ is satisfactorily small.

exercises

9.9.1. Write a *nonlinear* finite element program to compute the large bending of the tower in Fig.9.24(b). Use the Gauss band solver of the previous exercise in your Newton Raphson iterative scheme. Trace the displacement of the attacked node vs. the magnitude of the force pushing it.

9.9.2. Determine the stiffness, or lack of it, of the 9-rod symmetric and asymmetric, trusses of Fig.9.27, in plane and in space.

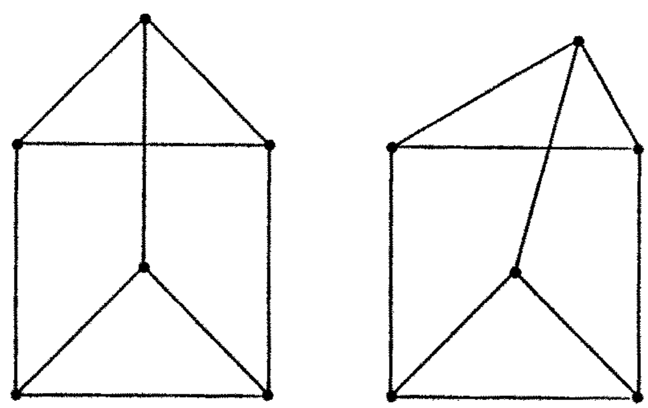


Fig 9.27

9.9.3. Determine the stiffness of a pyramidal truss with a rectangular base and triangular sides. It is 5—node, 8—bar structure.