1 Overview

Introduction

My research focuses on developing a global understanding of how complex systems change over time, and bridging the gap between what can be proven mathematically and what can be computed numerically.

Nonlinear differential equations are rarely explicitly solvable by hand. Instead of searching for arbitrary solutions, the dynamical systems viewpoint is to focus one's analysis on the qualitative behavior of **invariant sets**[1]. For example, while a periodic orbit's geometry may be sensitive to perturbations, its topology (eg being homeomorphic to a circle) is much more robust. With abstract theorems one can describe in great detail the dynamics on and around generic invariant sets. However for a specific differential equation, verifying the hypotheses of such a theorem often requires hard quantitative analysis.

I am particularly interested in **infinite dimensional dynamical systems**[2] and understanding their dynamics through a holistic study of a system's invariant sets. Computationally, this draws on a variety of numerical techniques from dynamical systems, partial differential equations, nonlinear optimization, and algebraic topology. Analytically, this often involves proving theorems with explicitly verifiable hypotheses (eg rather than assuming "*there exists some* $\epsilon > 0$", concretely quantifying how small ϵ must be). The impetus for this is not bookkeeping for bookkeeping's sake, but rather to build a complete picture of a complex system from disparate components.

For example, standard numerical methods can solve an initial value problem for an ODE and provide *local* error bounds at each step. However a *global* error bound on the final solution requires the cumulative error be quantified. This quickly becomes a nontrivial problem in chaotic systems, where arbitrarily close initial conditions will inevitably diverge, and the difficulties compound in partial differential equations where the phase space is infinite dimensional.

To that end, **validated numerics** have been developed to keep track of all the sources of error inherent to numerical calculations. To bridge the gap between numerics and a **computer assisted proof**, a problem must be translated into a list of the conditions that the computer can check. Most famously used to solve the four color theorem [2], computer assisted proofs have also been employed to great effect in dynamics, proving results such as the universality of the Feigenbaum constants [24] and Smale's 14^{th} problem on the nature of the Lorenz attractor [30].

Past Work

In my Ph.D. dissertation I solved a pair of longstanding conjectures about the global dynamics of a nonlinear delay differential equation (DDE) – where the rate of change depends nonlocally on both the past and the present (cf §4). First studied in 1955, Wright's equation has come to be known as a canonical example of a nonlinear scalar DDE. The conjectures by Wright and Jones claimed that for parameters $\alpha < \frac{\pi}{2}$ all initial data is attracted to 0, and for $\alpha > \frac{\pi}{2}$ almost all initial data is attracted to a unique periodic orbit [21, 34]. Building upon prior work, we used a combination of detailed bifurcation analysis and computer-assisted-proofs to prove these conjectures [13, 17, 31].

My recent research has focused on the dynamics of PDEs, both dissipative and dispersive (cf §2). More specifically, my work has focused on proving existence and stability of certain coherent structures [5,6,14], characterizing how trajectories transition from one coherent structure to another [18, 19], and studying the behavior of blowup solutions [14, 19, 29]. In a series of papers I have been

[1] An invariant set is a closed subsystem within the dynamical system; i.e. all points in the set must stay in set under the dynamics. Examples include steady states, periodic orbits, stable/unstable manifolds, and chaotic attractors.

[2] An infinite dimensional dynamical system is a dynamical system whose phase space is infinite dimensional (such as a PDE describing the time evolution of functions), rather than finite dimensional (such as an ODE describing the time evolution of points in $\mathbb{R}^n$).

studying PDEs with vortex-stretching type nonlinearities, and have demonstrated a rich zoo of dynamical behavior and blowup solutions (cf §2.2-§2.4).

Differential equations can be extremely useful in describing complex phenomenon, however it is usually impossible to completely understand them using analytical methods. Often a coarser but computationally tractable description is needed (cf §3). In recent years computational topology has become widely recognized as an important tool for quantifying complex structures, and in [16, 20] we applied persistent homology to study spatial patterns and self-similar structures. Biological systems are prone to generate complex chaotic activity, yet many experimental observations reported robustness of biological systems to changes in external or internal conditions, and in [15] we study how biological function can be maintained or disrupted in chaotic slow-fast systems.

Current & Future Work

In my future research, I will continue to study the dynamics of PDEs using analytic and computer-assisted-proof techniques to bridge the gap between numerical evidence and mathematical certainty. Towards this goal, my current projects focus on proving the existence and stability of a variety of coherent structures (eg steady states, pulses, blowup profiles). Beyond analyzing each piece in isolation, my future work will study how trajectories travel from invariant set to another, and develop a global understanding of the dynamics.

Using the Swift-Hohenberg equation as an exemplary system of pattern formation, current projects focus on computing spectral stability of pulses (unbounded domains) and connecting orbits between unstable steady states (bounded domains). In another project I am studying self-similar blowup in a nonlinear Schrödinger equation with a vortex-stretching type nonlinearity. More broadly I plan to use persistent homology to empirically analyze self-similar blowup in PDEs.

One strength of validated numerics is their potential to yield non-perturbative results, even in scenarios where global existence or generic well-posedness may not be guaranteed [7]. One such example are the incompressible Navier-Stokes equations, which describe the motion of viscous fluids and for which the global existence of solutions in 3D is famously unknown [32, 33]. The case of axisymmetric flow with swirl contains many of the essential difficulties of the problem, and recent numerical work has uncovered initial data yielding nearly singular behavior [12]. In current work I am developing validated numerics to study steady states of the forced NSE in this geometry. Beyond the intrinsic interest of the problem, this work will develop validated spectral methods for more complicated (ie non-toroidal) geometries. In future work I plan to study how these steady states continue, and bifurcate to produce more complex dynamics.

A fundamental innovation in my PhD thesis was developing a numerical method which will exhaustively search through an infinite-dimensional phase space, making sure that no other attractors could exist [13, 17]. This Fourier/spectral methodology is by no means limited to DDEs. An exciting student research project would be to develop computer-assisted-proofs for an exact count of the number of equilibria of nonlinear parabolic PDEs on finite intervals and eventually multidimensional domains. Such an enumeration would be a powerful tool to analyze systems with a global attractor (eg Swift-Hohenberg equation, forced Navier-Stokes equations [28]).

2 Dynamics of Partial Differential Equations

Understanding the long term behavior of solutions is fundamental to the study of evolutionary equations. This begins with questions of existence, and whether local existence can be extended globally. Of the solutions which exist globally, coherent structures (such as equilibria, traveling waves and periodic orbits) serve as emblematic examples of how solutions to a PDE may behave. My recent research has largely focused on the dynamics of PDEs, in particular proving existence and stability of certain coherent structures, characterizing how trajectories transition from one coherent structure to another, as well as studying the behavior of blowup solutions.

2.1 Stability of coherent structures

The local stability of coherent structures helps to inform us as to what type of phenomena are generically observable: while stable structures are robust and attract nearby trajectories, unstable objects repel solutions and are harder to detect. Nevertheless unstable objects are critically important in guiding the transient behavior of a system, and understanding to which particular stable state solutions will be attracted, see Fig. 1.

For a PDE on a finite domain one can use a Galerkin projection to approximate the dynamics and calculate the stability of equilibria. As the projection dimension approaches infinity, standard numerics can guarantee that solutions of the finite dimensional system will converge to solutions of the infinite dimensional system. However for a fixed, finite dimensional Galerkin projection, a natural question is: *How close are the approximate solutions to the true solutions*? In [6, 18, 19, 29] I have been exploring how to answer this question in a variety of scenarios, and using computer-assisted-proofs to provide explicit error bounds.

For example, in [6] with JB van den Berg (VU Amsterdam) and J Mireles James (Florida Atlantic University), we present a rigorous computational method for approximating infinite dimensional stable manifolds of non-trivial equilibria for parabolic PDEs. Our approach combines the parameterization method – which can provide high order approximations of finite dimensional manifolds with validated error bounds – together with the Lyapunov-Perron method – which is a

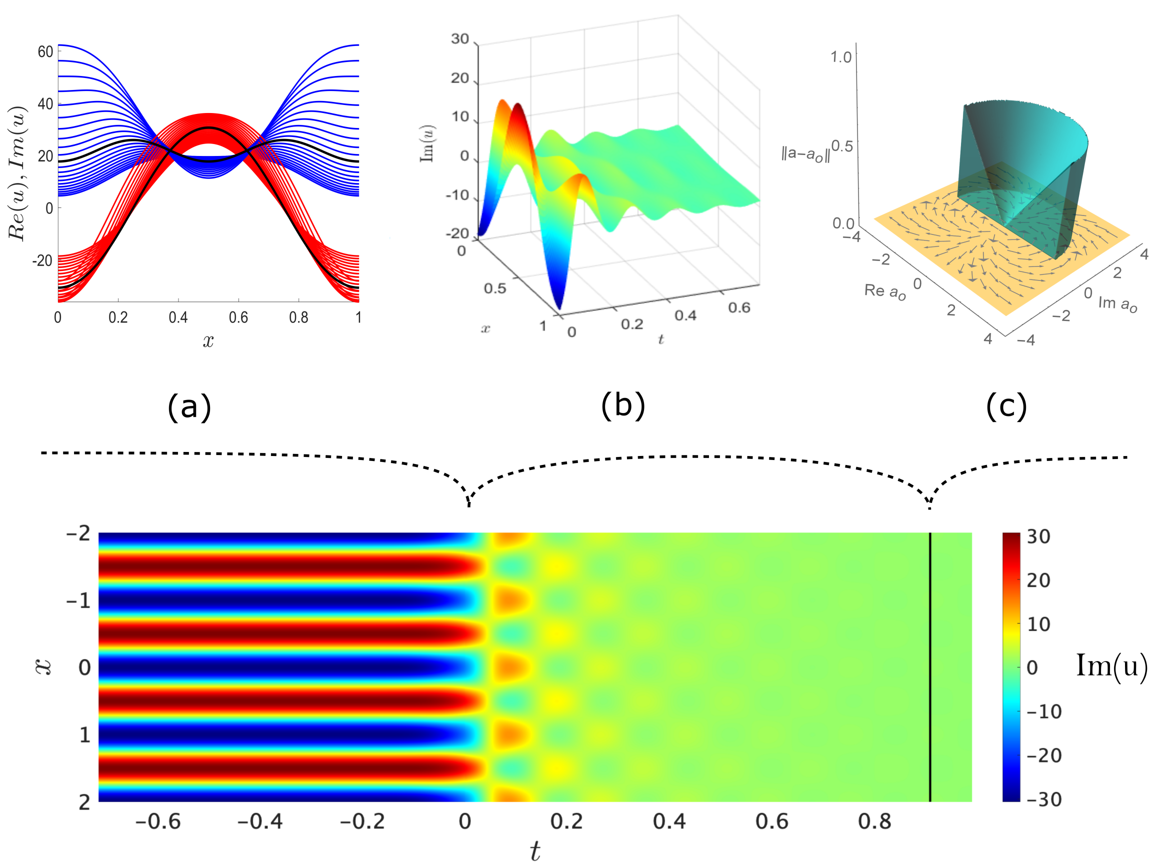

Fig. 1: A connecting orbit to the nonlinear Schrödinger equation $iu_t = u_{xx} + u^2$ with $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, cf [18]. (a) Real and imaginary components of a nontrivial equilibrium, with two trajectories on its unstable manifold; (b) Rigorous integration of an endpoint from the unstable manifold; (c) A trapping cone in Fourier space of points converging to the center-type 0-equilibrium.

powerful technique for proving the existence of (potentially infinite dimensional) invariant manifolds. As an example, we apply this technique to approximate the stable manifold associated with unstable nonhomogeneous equilibria for the Swift-Hohenberg equation on a finite interval.

For PDEs on unbounded domains Galerkin methods run into difficulties. Recent work on the Maslov index has extended classical results from Sturm-Liouville theory to a much more general setting, thus allowing for spectral stability of nonlinear waves in a variety of contexts to be determined by counting conjugate points. With M Beck (Boston University) we have developed a framework for the computation of conjugate points using rigorous numerics [5]. We apply our method to a parameter-dependent system of bistable equations and show that there exist both stable and unstable standing fronts. In comparison with rigorous numerical methods to compute stability using the Evans function [3], our results suggest that counting conjugate points is much more efficient. With PhD student Hannah Pieper (Boston University), we are extending this methodology to 4^{th} order systems such as the Swift-Hohenberg equation.

2.2 Global dynamics in a toy vortex-stretching model

In a series of work [14, 18, 19, 29] I have been studying the family of complex valued PDEs:

$$u_t = e^{i\theta}\left(u_{xx} + u^2\right), \qquad\qquad x \in \mathbb{T} \equiv \mathbb{R}/\tfrac{2\pi}{\omega}\mathbb{Z}. \tag{1}$$

for parameters $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. When $\theta = 0$, then (1) is equivalent to the viscous Constantin-Lax-Majda model of hydrodynamic vortex stretching [9]. When $\theta = \pm\frac{\pi}{4}$ the equation resembles the complex Ginzburg-Landau equation, and when $\theta = \pm\frac{\pi}{2}$ the equation becomes a nonlinear Schrödinger equation. While it is well known that solutions to (1) can blowup, the equation exhibits a rich zoo of dynamical behavior which had scarcely been explored.

Together with JP Lessard (McGill University) and A Takayasu (University of Tsukuba), we use computer assisted proofs to demonstrate existence of nontrivial equilibria, and heteroclinic orbits between these nontrivial equilibria and 0, see Fig. 1.

Theorem 1 ([18,19]). *The PDE* (1) *has at least two non-trivial, linearly unstable equilibria. For* $\theta \in \{0, \pm\frac{\pi}{4}, \pm\frac{\pi}{2}\}$*, and for each equilibrium* $\tilde{u}$ *there exists a heteroclinic orbit traveling from* $\tilde{u}$ *to* 0.

Connecting orbits provide a road map for how a dynamical system transitions between its various fixed points and periodic orbits. Certain kinds of connecting orbits, such as homoclinics from a periodic orbit to itself, can be used to prove the existence of mathematical chaos. In the temporal dynamics of a PDE, a connecting orbit between two nonhomogeneous equilibria describes how perturbations to an unstable equilibrium unfold, and to which stable equilibrium the perturbed state will be attracted.

In future work I plan to develop validated numerics for computing saddle-to-saddle connecting orbits in the Swift-Hohenberg equation, which will build upon our work in [6] computing infinite dimensional stable manifolds. Longer term goals include constructing a computer assisted proof of chaos in a PDE via a homoclinic tangle, and computing connecting orbits in strongly indefinite problems motivated from multidimensional traveling waves.

In addition to developing methods to analyze more complicated dynamical behavior, I am also extending our current methodology to study a broader class of PDEs. In current work I am developing validated numerics to study steady states in the forced incompressible 3D Navier-Stokes equation with axisymmetric symmetry. Beyond the intrinsic interest of studying the NSE in a geometry where global existence is unknown, this work will develop validated spectral methods for more complicated (ie non-toroidal) geometries. In future work I plan to study how these steady states continue, and bifurcate to produce more complex dynamics.

2.3 Nonconservative nonlinear Schrödinger equations

More generally, the NLS case ($\theta = -\frac{\pi}{2}$) of (1) may be considered with higher power nonlinearities $p \geq 2$ and higher spatial dimensions $d \geq 1$:

$$iu_t = \triangle u + u^p, \qquad\qquad x \in \mathbb{T}^d \equiv \mathbb{R}^d / \tfrac{2\pi}{\omega}\mathbb{Z}^d. \tag{2}$$

This NLS does not have gauge invariance, $(e^{i\theta}u)^p \neq e^{i\theta}u^p$ for generic $\theta \in \mathbb{R}$, and it does not admit a natural Hamiltonian structure. In the spatially homogeneous dynamics of (2), the origin is foliated by homoclinic solutions with the exception of some finite time blowup solutions. These homoclinic solutions in fact extend to an open set of initial data, which allows us to show that (2) is nonconservative.

Theorem 2 ([18])**.** *There exists an open set of complex initial data with summable Fourier coefficients whose solutions to* (2) *are homoclinic orbits, limiting to* 0 *in forward/backward time.*

Corollary 3 ([18])**.** *The only analytic functionals conserved under* (2) *are constant.*

Despite the nonconservative nature of this system, I show in [14] that it surprisingly exhibits another facet of integrability[3]. Namely, the space of positive Fourier modes form an invariant subspace, wherein a solution's Fourier coefficients may be explicitly solved for order by order, akin to the cubic Szegő equation [11] or the Constantin-Lax-Majda equation [9, 26]. Moreover, I show that small initial data to (2) will yield quasiperiodic solutions, all with fixed frequencies $\{\omega_j^2/2\pi\}_{j=1}^d$ determined by the geometry of the torus $\mathbb{T}^d$, cf Fig. 2. In future work I plan to explore how this facet of integrability may be connected to an underlying singular symplectic structure [25].

[3] Integrability is associated with having a maximal number of conserved quantities and being able to explicitly construct solutions by definite 'integrals'.

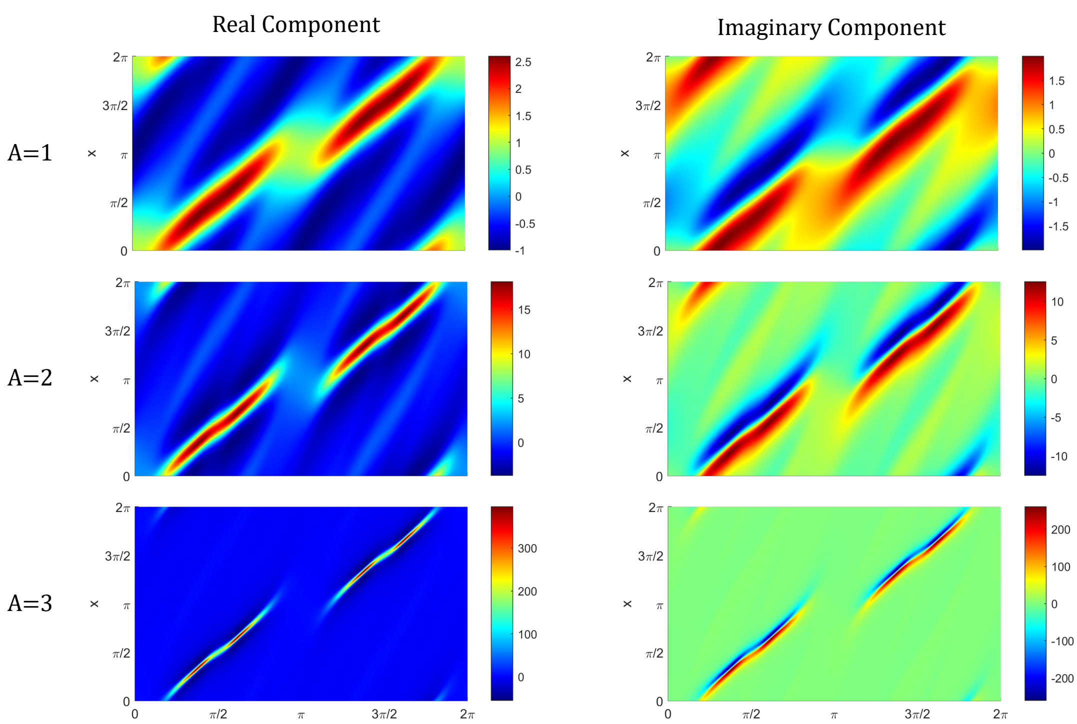

Fig. 2: Solutions to the NLS $iu_t = u_{xx} + u^2$ with initial data $u_0(x) = Ae^{ix}$. If $|A| \leq 3$ the solution is 2π-periodic, however if $|A| \geq 6$ the solution blows up in finite time, cf [14].

2.4 Unbounded growth of solutions in finite/infinite time

It is often the case that global existence may be proved for small initial data, but the argument breaks down for large data. One is left to wonder: *Can some other proof techniques show global existence for large data, or is there a counter-example (eg finite time blowup)?*

Using (1) as a toy model I have explored answering this question using several different techniques. In the NLS case ($\theta = \pm\pi/2$), it turns out that monochromatic inital data will yield periodic solutions if the data is small (cf Fig. 2) but will blow up if the data is large [14]. In the heat and CGL case ($\theta = 0, \pm\pi/4$) there do not appear to be any periodic orbits. However by using a forcing argument, we prove there exists solutions limiting to the non-trivial equilibria in backwards time, and are unbounded in forward time [19].

In [29] with A Takayasu (University of Tsukuba), JP Lessard (McGill University), and H Okamoto (Gakushuin University), we study the PDE $u_t = u_{xx} + u^2$ for $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and the behavior of solutions when continued past their blowup time. It is well-known that dissipative PDEs have solutions that will exist not only for some interval $t \in [0, \epsilon)$, but moreover the solution $u(t)$ can be analytically continued for complex values of $t \in \mathbb{C}$. By solving the PDE, with a validated numerical integrator we developed, along a contour in the complex plane of time, and showing that the contours in the upper and lower halves of the complex plane yield different solutions, we are able to prove that a branching singularity occurs.

In current work, I am pursuing a conjecture by Cho et al. [8], that the NLS case ($\theta = \pm\pi/2$) of (1) is globally well posed for all real initial data. In [18] we proved global existence for close-to-constant real initial data. However, my recent numerical simulations indicate that there is a meagre set of real initial data that does blowup. To rigorously prove this to be the case, I aim to combine validated numerics with a center-stable manifold analysis of the self-similar dynamics.

3 Coarse-grained analysis of complex systems

3.1 Computational algebraic topology

Partial differential equations can be extremely useful in describing patterns in biological and physical systems. However these patterns can be quite complicated, exhibiting distinct structures at different spatial/temporal scales, and it is usually impossible to completely understand them using analytical methods. Often a coarser but computationally tractable description is needed. In recent years computational topology has become widely recognized as an important tool for quantifying complex structures.

Persistent homology is an algebraic tool that provides a mathematical framework for analyzing the multi-scale structures frequently observed in nature. For a given point cloud, one may construct a nested sequence of topological spaces by growing ϵ-balls about each point, see Fig. 3. The 0-dimensional persistent homology tracks when connected components first appear, and later merge together; the 1-dimensional persistent homology tracks when loops in the space appear and disappear.

Long persistence intervals are generally considered to correspond to important topological features, whereas short intervals are considered to be noise. That is, as the number of points n increases, the important persistence intervals will stabilize whereas the average length of the noisy intervals will decrease. However if a point cloud is sampled from a d-dimensional Lebesgue measure, the summed-length of all 0-dimensional persistence intervals will grow in proportion with $n^{\frac{d-1}{d}}$.

In fact, a fractional dimension can be defined for a measure in terms of the asymptotic growth of the totaled persistence intervals of point samples [27]. In [20] with B Schweinhart (George Mason University) we implement an algorithm to estimate the i-dimensional persistent homology dimension ($i = 0, 1, 2$) to study self-similar fractals, chaotic attractors, and an empirical dataset of

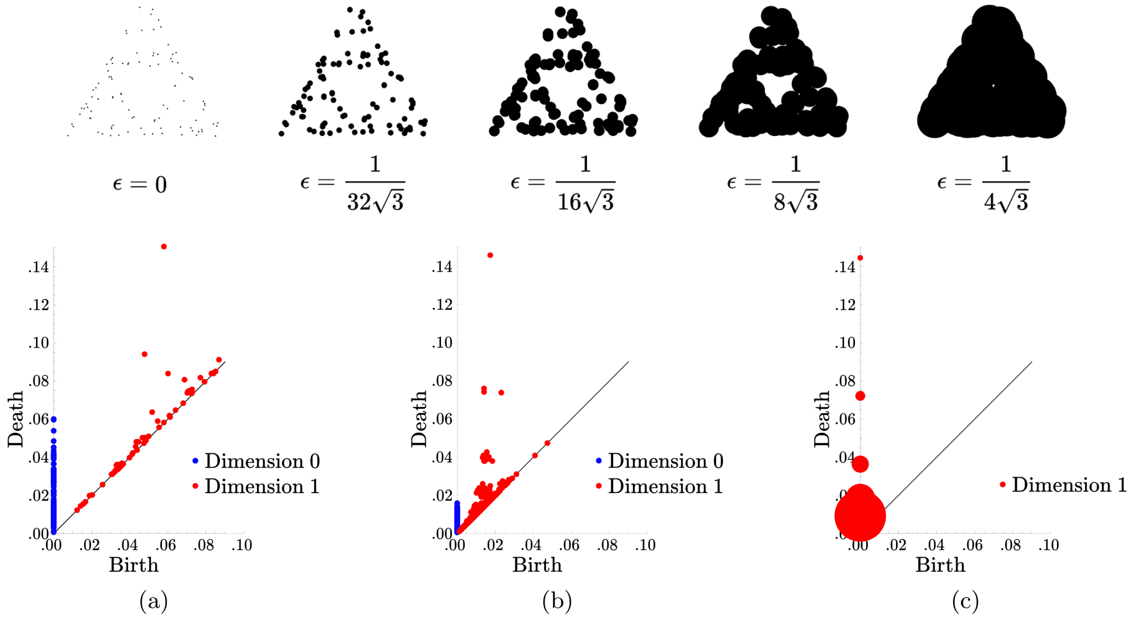

Fig. 3: Top: ϵ-neighborhoods about a sample of 100 points from the Sierpinski triangle. Bottom: Persistence diagrams for (a) 100 point sample from the Sierpinski triangle, (b) 1000 points from the Sierpinski triangle and (c) the Sierpinski triangle itself (where the area of a dot is proportional to the number of persistent homology intervals with corresponding birth and death time).

earthquake hypocenters.

Often when a PDE blows up, a solution's magnitude and spatial dependence become increasingly singular according to a power law as the blowup time is approached. However by making a change of coordinates into self-similarity variables, the solution can be renormalized. In these new coordinates, a blowup solution may be understood as limiting towards a fixed point, limit cycle or even a strange attractor [10]. In future work I plan to use persistent homology to empirically study self-similar blowup in PDEs. This topological tool is particularly well suited for the task, as a function's persistent homology is invariant under continuous deformations of its domain, providing a robust lens to analyze complex spatio-temporal behavior [23].

3.2 Global bifurcations and chaos in slow-fast systems

Nonlinear systems in interaction, often used to describe biological systems, are prone to generate complex chaotic activity, yet many experimental observations reported robustness of biological systems to changes in external or internal conditions. In [15], with E Sander (George Mason University), S Kedia (Brandeis University), and J Touboul (Brandeis University), we study how biological function can be maintained or disrupted in chaotic slow-fast systems. In particular we propose a refinement of the notion of chaos that reconciles chaos and biological robustness in chaotic systems with multiple timescales.

We find that systems displaying relaxation cycles going through strange attractors do generate chaotic dynamics that are regular at macroscopic timescales, thus consistent with physiological function. However, this relative regularity breaks down through a universal global bifurcation, beyond which the system generates erratic activity also at slow timescales. Our manuscript focuses on the analysis of an exemplar system describing nerve cell activity and data in a crustacean central pattern generator. Beyond this example, we show that the passage of slow relaxation cycles through a strange attractor crises is a universal mechanism for the transition in such dynamics, and future work will investigate such crisis bifurcations in more realistic neuronal models [1, 22].

4 Wright's Conjecture on a Nonlinear Delay Differential Equation

In my thesis I proved two half-century old conjectures concerning the delay differential equation known as Wright's equation:

$$x'(t) = -\alpha\big(e^{x(t-1)} - 1\big). \tag{3}$$

First studied in 1955 as a heuristic model of the distribution of primes [34], Wright's equation has come to be known as a canonical example of a nonlinear scalar delay differential equation (DDE). As with partial differential equations, the initial data for DDEs are functions. In Wright's seminal work he showed that if $\alpha \leq \frac{3}{2}$ then the equilibrium solution $x \equiv 0$ is the global attractor, and made the following conjecture:

Theorem 4 (Wright's Conjecture, 1955)**.** *For every* $0 < \alpha \leq \frac{\pi}{2}$ *the equilibrium solution* $x \equiv 0$ *to* (3) *is globally attractive.*

In 1962 Jones [21] proved that for $\alpha > \frac{\pi}{2}$ there exists at least one *slowly oscillating periodic solution (SOPS)* to Wright's equation. That is, a periodic solution $x : \mathbb{R} \to \mathbb{R}$ such that it is positive for at least the length of the time delay, and then negative for at least the length of the time delay. Based on numerical simulations Jones made the following conjecture:

Theorem 5 (Jones' Conjecture, 1962)**.** *For every* $\alpha > \frac{\pi}{2}$ *there is a unique slowly oscillating periodic solution to* (3)*.*

I proved both of these conjectures in a trio of papers [13,17,31]. Prior work had proved Wright's conjecture for $\alpha < \frac{\pi}{2} - 2 \times 10^{-4}$ via a computer assisted proof which took months of CPU time, and as authors mention, "*substantial improvement of the theoretical part of the present proof is needed to prove Wright's conjecture fully*" [4]. Hopf bifurcations are canonically analyzed with the method of normal forms, which transforms a given equation into a simpler expression having the same qualitative behavior as the original equation. By an implicit-function-theorem type argument, this transformation is valid in some neighborhood of the bifurcation. However, the proof does not offer any insight into the size of this neighborhood. In [31] with JB van den Berg (VU Amsterdam) we develop an explicit description of a neighborhood wherein the only periodic solutions are those originating from the Hopf bifurcation. The main result of this analysis is the resolution of Wright's conjecture.

In 1991 Xie [35] proved Jones' conjecture for $\alpha \geq 5.67$. He accomplished this by first showing that there is a unique slowly oscillating periodic solution to (3) if and only if every SOPS is asymptotically stable. By using asymptotic estimates of SOPS for large α, Xie was able to estimate their Floquet multipliers and prove that all SOPS had to be stable. However, at $\alpha = 5.67$ these asymptotic estimates break down.

In [17] with JP Lessard (McGill University) and K Mischaikow (Rutgers University) we used the same basic method as Xie, however we replace the asymptotic estimates with validated numerics. We use a branch and bound algorithm to develop pointwise estimates on all the possible SOPS to Wright's equation and then bound their Floquet multipliers. Using these two main steps, we generate a computer assisted proof for $\alpha \in [1.9, 6.0]$

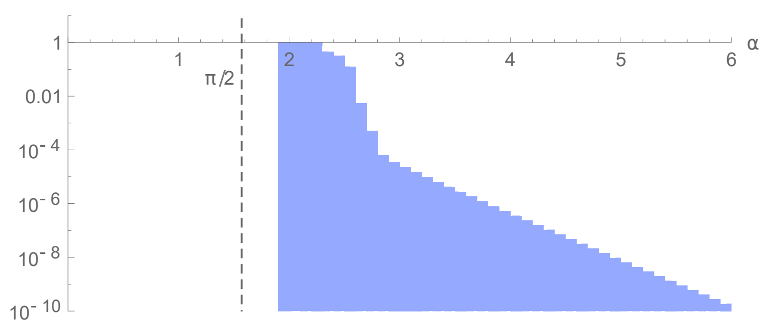

Fig. 4: A strict upper bound on the modulus of the Floquet multipliers for SOPS to Wright's equation [17].

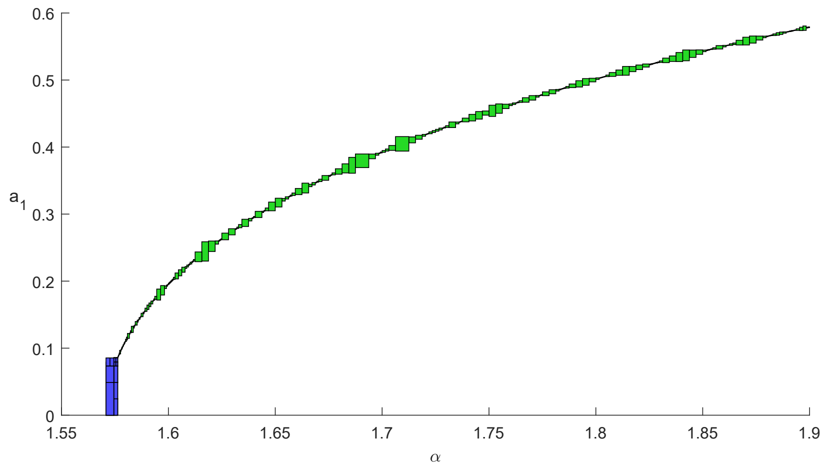

Fig. 5: The first Fourier coefficient of SOPS to Wright's equation, which has a Hopf bifurcation at $\alpha = \frac{\pi}{2}$. Uniqueness of the SOPS with respect to α is proved for the green cubes in [13] and for the blue cubes in [31].

that all SOPS to Wright's equation are asymptotically stable, and thereby unique up to translation.

I finished the proof to Jones' conjecture in [13], proving there is a unique SOPS for $\alpha \in (\frac{\pi}{2}, 1.9]$. While previous work [31] showed that there are no folds in the principal branch of periodic orbits this did not rule out the possibility of isolas, that is SOPS far away from the principal branch. To rule out the existence of these isolas, we recast the problem of studying periodic solutions to (3) as the problem of finding zeros of a functional defined on a space of Fourier coefficients, and again employed an infinite dimensional branch and bound algorithm.

This methodology for exhaustively searching through an infinite-dimensional phase space to find *all* of the solutions is by no means limited to DDEs. An exciting student research project would be to develop computer-assisted-proofs for an exact count of the number of equilibria of nonlinear parabolic PDEs on compact domains. While obtaining an initial enclosure of all solutions would be problem specific, the essential component is to apply elliptic bootstrapping and then perform the branch and bound algorithm from [13]. Such an enumeration would be a powerful tool to analyze systems with a global attractor (eg Swift-Hohenberg equation, forced Navier-Stokes equations [28]).

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