Reaction-Diffusion Equations In Narrow Tubes and Wave Front Propagation

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Part I

Introduction and Real Life Examples

Definitions

Wave propagation is any of the ways in which waves travel through a medium.

Wavefront is the locus (a line, or, in a wave propagating in 3 dimensions, a surface) of points having the same phase.

Examples

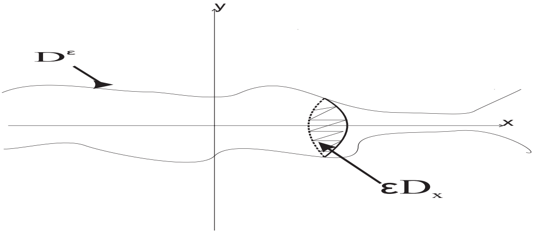
- Radio propagation and electromagnetic waves.
- Signal transmission and fiber optics.
- Surface waves in water.
- Electrical activity in membranes of living organisms.
- Nano-tubes in nano-technology.
- In combustion theory a wave describes how solid fuel or gas is burnt as the flame front passes through a long, **narrow domain**.

Part II

Description of the Problem and Main Result

Description of the problem.

Let $D^{\epsilon}=\{(x,y): x\in\mathbb{R}^{1}, y\epsilon^{-1}\in D_{x}\subset\mathbb{R}^{2}\}\subset\mathbb{R}^{3}$ be a thin tube of width $\epsilon\ll 1$ in $\mathbb{R}^{3}$.



Our aim is to study reaction-diffusion equations in the narrow tube D^{ϵ}. It is of particular interest to mathematically explain the **effect that the width of the tube** may have in the propagation of the wave front.

Description of the problem.

We model the flow of the quantity of interest by $u^\epsilon = u^\epsilon(t,x,y)$ where u^ϵ is the solution to the following P.D.E.

$$\begin{aligned} u^\epsilon_t &= \frac{1}{2}\triangle u^\epsilon, && \text{in } (0,T)\times D^\epsilon \\ u^\epsilon(0,x,y) &= f(x), && \text{on } \{0\}\times D^\epsilon \\ \frac{\partial u^\epsilon}{\partial \gamma^\epsilon} &= -\epsilon c(x,y,u^\epsilon)u^\epsilon, && \text{on } (0,T)\times \partial D^\epsilon, \end{aligned} \tag{1}$$

where γ^ϵ is the inward unit normal to ∂D^ϵ and $\triangle u^\epsilon = u^\epsilon_{xx} + u^\epsilon_{yy}$. *The goal is to examine the behavior of the function $u^\epsilon(t,x,y)$ as $\epsilon \downarrow 0$.*

Motivation.

In the theory of excitable media:

- Each point in $\mathbb{R}^3$ is allowed to attain two states: it is either excited or non-excited (**threshold** effect).
- The excitation (i.e. the set of excited points) expands with increasing time. Each point in $\mathbb{R}^3$ which is reached by the excitation at time t becomes immediately excited and remains in this state for ever. Beginning with the moment t, the point in $\mathbb{R}^3$ itself serves as a source for the further propagation of excitation.

Questions to answer...

More specifically the questions that we ask, are the following:

1. Is it possible to find a set Q so that

$$\lim_{\epsilon \downarrow 0} u^{\epsilon}(t,x,y) = \begin{cases} 1, & (t,x) \in Q \\ 0, & (t,x) \notin Q \end{cases} \tag{2}$$

2. What are the properties of the set Q ?
3. How does the wave front propagates? Are jumps of the wave front possible?
4. How does the volume of the cross-sections D_x affect the propagation of the wave front ?

answers...

Assumptions: Slowly Changing Media, KPP Nonlinearity.

We assume

- The functions $c(\cdot, 0, u)$, $f(\cdot)$ and the cross-section D_x change slowly in x. This means, for example, that $f(\cdot) = f(\delta x)$ for $0 < \delta \ll 1$.
- The nonlinear boundary term $c(x, y, u)$, is of K-P-P type for $y = 0$:
    - $c(x, 0, u)$ is positive for $u < 1$
    - $c(x, 0, u)$ negative for $u > 1$
    - $c(x) = c(x, 0, 0) = \max_{0 \le u \le 1} c(x, 0, u)$.

Wave Front in Slowly Changing Media.

Then one can prove (K.S. M.F. 2008) that

$$Q = \{(t,x) : W(t,x) > 0\}.$$

Here W is a function of (t,x) that is the solution to some Hamilton-Jacobi-Bellman equation.

So

$$\lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} u^{\epsilon}\left(\frac{t}{\delta}, \frac{x}{\delta}, y\right) = \begin{cases} 1, & W(t,x) > 0 \\ 0, & W(t,x) < 0 \end{cases} \quad (3)$$

The equation $W(t,x) = 0$ defines the position of the interface (wavefront) between areas where u^{ϵ} (for $\epsilon > 0$ small enough) is close to 0 and to 1.

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Sketch of the proof.

Step 1. *A-priori bounds for the solution:*

There is a constant C, independent of ϵ, and an open set $I \subset (0,1)$ such that for any $a \in I$:

$$\overline{\|u^{\epsilon}\|}_{D^{\epsilon},T,1+a} + \|D^{2}u^{\epsilon}\|_{V_{T}^{\epsilon}} \leq C. \tag{4}$$

Here $\overline{\|u^{\epsilon}\|}_{D^{\epsilon},T,1+a} = \|u\|_{\overline{D}_{T}^{\epsilon},a} + \|u_{t}\|_{\overline{D}_{T}^{\epsilon}} + \|Du\|_{(0,T)\times\overline{D}^{\epsilon}}$.

Sketch of the proof.

Step 2. *Feynmann-Kac Formula:*

$$u^\epsilon(t,x,y) = E_{x,y} f(X_t^\epsilon) \exp\left[\int_0^t \epsilon c(X_s^\epsilon, Y_s^\epsilon, u^\epsilon(t-s, X_s^\epsilon, Y_s^\epsilon)) dL_s^\epsilon\right] \quad (5)$$

Here $(X_t^\epsilon, Y_t^\epsilon)$ is the Wiener process in D^ϵ with reflection on ∂D^ϵ and L_t^ϵ is the local time for this process. Its trajectories are described by the S.D.E.:

$$\begin{aligned} X_t^\epsilon &= x + W_t^1 + \int_0^t \gamma_1^\epsilon(X_s^\epsilon, Y_s^\epsilon) dL_s^\epsilon \\ Y_t^\epsilon &= y + W_t^2 + \int_0^t \gamma_2^\epsilon(X_s^\epsilon, Y_s^\epsilon) dL_s^\epsilon. \end{aligned} \quad (6)$$

Sketch of the proof.

Step 3. *Convergence of Underlying Stochastic Process:*
Let X_t be the solution of the stochastic differential equation

$$X_t = x + W_t^1 + \int_0^t \frac{1}{2}\nabla(\log V(X_s))ds. \tag{7}$$

where $V(x)$ is the volume of D_x.
Let $H(x,y)$ be a given smooth function and define
$Q(x) = \frac{1}{V(x)}\int_{\partial D_x} H(x,y)dS_x$. Then for any $T>0$ and as $\epsilon \downarrow 0$:

$$\sup_{t\leq T} \quad E|X_t^\epsilon - X_t|^2 \to 0.$$

$$\sup_{t\leq T} \quad E\left|\int_0^t \frac{1}{2}Q(X_s^\epsilon)ds - \int_0^t \epsilon H(X_s^\epsilon, Y_s^\epsilon/\epsilon)|\gamma_2^\epsilon(X_s^\epsilon, Y_s^\epsilon)|dL_s^\epsilon\right|^2 \to 0$$

Sketch of the proof.

Step 4. *Limit of u^{ϵ}:*

$$u^{\epsilon}(t,x,y) \to u(t,x) \text{ as } \epsilon \to 0, \text{ uniformly in any compact subset of } \mathbb{R}_{+} \times \mathbb{R}^{n} \times \mathbb{R}^{m},$$

where $u(t,x)$ is the solution to:

$$u(t,x) = E_{x} f(X_{t}) \exp\left[\int_{0}^{t} \frac{S(X_{s})}{V(X_{s})} c(X_{s}, 0, u(t-s, X_{s})) ds\right]. \quad (8)$$

By Feynmann-Kac formula it satisfies

$$\begin{aligned} u_{t} &= \frac{1}{2}\triangle_{x} u + \frac{1}{2}\nabla(\log V(x))\nabla_{x} u + \frac{S(x)}{V(x)} c(x,0,u)u \\ u(0,x) &= f(x). \end{aligned} \quad (9)$$

Here $V(x)$ is the volume of D_{x} and $S(x)$ is the surface area of ∂D_{x}.

Sketch of the proof.

Step 5. *Limit of* $u^{\delta}(t,x) = u(t/\delta, x/\delta)$*:*

Then under certain conditions (M.F.) we have:

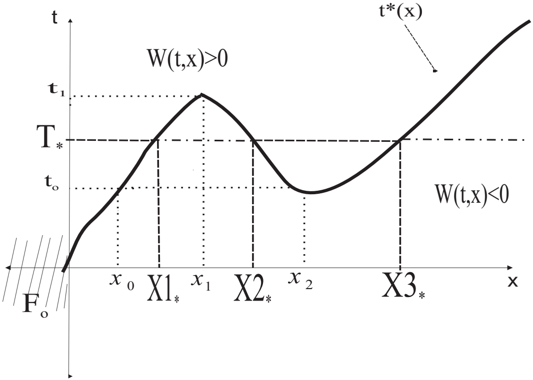
$$\lim_{\delta \downarrow 0} u^{\delta}(t,x) = \begin{cases} 1, & W(t,x) > 0 \\ 0, & W(t,x) < 0 \end{cases} \tag{10}$$

So putting things together we have

$$\lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} u^{\epsilon}(\frac{t}{\delta}, \frac{x}{\delta}, y) = \begin{cases} 1, & W(t,x) > 0 \\ 0, & W(t,x) < 0 \end{cases} \tag{11}$$

When does the Wave Front have Jumps ?

Let $t^* = t^*(x)$ be such that $W(t^*, x) = 0$.



- The wavefront jumps from x_o to x_2 at time t_o.
- For $t = T_*$ the points $x \in (0, X1_*) \cup (X2_*, X3_*)$ **are excited** whereas the points $x \in (X1_*, X2_*)$ **are not excited**.

When does the Wave Front have Jumps ?

Let $\bar{c}(x) = \frac{S(x)}{V(x)} c(x,0,0)$, where $S(x)$ and $V(x)$ are the surface area and the volume of the cross-sections D_x respectively.

- If $\bar{c}(x)$ **increases rapidly** at some point x, then $t^* = t^*(x)$ is as in the previous figure (K.S, M.F 2008).

Special case:

If $c(x,0,0)$ is constant

$\Downarrow$

Jumps occur at places where $\frac{S(x)}{V(x)}$ increases rapidly

$\Updownarrow$

Jumps occur at places where the tube D^1 becomes thinner (*at least when the tube D^1 retains its shape as x increases*).

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References.

- S.N. Ethier, T.G. Kurtz, Markov processes: Characterization and Convergence, Wiley, New York, 1986.
- M. Freidlin and K. Spiliopoulos, 2008, "Reaction Diffusion Equations with non-linear boundary conditions in narrow domains", submitted to the journal of Asymptotic Analysis.
- M. Freidlin, Functional Integration and Partial Diffferential Equations, Princeton University Press, 1985.
- M. Freidlin, Wave Front Propagation for KPP-Type Equations, Survey in Applied Mathematics, 2 (1995), pp. 1-62.

References.

- M. Freidlin, Markov Processes and Differential Equations: Asymptotic Problems, Birkhäuser Verlang, 1996.
- M. Freidlin, Coupled Reaction Diffusion Equations, Annals of Probability, Vol. 19, No. 1 (1991), pp. 29-57. M.I. Freidlin, A.D. Wentzell, Random Perturbations of Dynamical Systems, Second Edition, Springer, 1998.
- A. Friedman, Partial Differential Equations of Parabolic Type, Prentice Hall, 1964.
- J. Gärtner, Bistable Reaction-Diffusion Equations and Excitable Media, Mathematiche Nachrichten, Vol. 112 (1983), pp. 125-152.

References.

- Peter Grindrod, The Theory and Applications of Reaction-Diffusion Equations, Patterns and Waves, Second edition, Oxford press, 1996.
- I.Karatzas, S.E.Shreve, Brownian Motion and Stochastic Calculus, Second edition, Springer, 1994.
- A. Kolmogorov, I. Petrovskii, N. Piskunov, *É*tude de l′èquation de la diffusion avec croissence de la matière et son application a un problème biologique, Moscov University Bull. Math., Vol. 1 (1937), pp. 1-25.
- J. Nolen, J. Xin, KPP Fronts in a One-Dimensional Random Drift, (2007), preprint.

THANK YOU!!!!!