Homogenization in PDE's and in Stochastic Processes: An Introduction.

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Let f be a smooth function, D be a bounded smooth function that is periodic with period 1 (i.e. $D(x) = D(x+1)$) and $\epsilon > 0$.

Consider the second order parabolic PDE:

$$\begin{aligned} u^\epsilon_t(t,x) &= L^\epsilon u^\epsilon(t,x) \\ u^\epsilon(0,x) &= f(x), \quad (t,x) \in (0,\infty) \times \mathbb{R} \end{aligned} \tag{1}$$

where for $u \in \mathcal{C}^2(\mathbb{R})$ we denote:

$$L^\epsilon u = \frac{1}{2\epsilon} D'(x/\epsilon) u_x(t,x) + \frac{1}{2} D(x/\epsilon) u_{xx}(t,x) \tag{2}$$

It is easy to see that

$$L^\epsilon u = \frac{1}{2}\frac{d}{dx}(D(x/\epsilon)\frac{du}{dx}).$$

We say that the elliptic operator (2) can be written in divergence form.

Our goal is to consider the limit as $\epsilon \downarrow 0$ of the solution u^ϵ to (1). The approach will be based on probabilistic methods.

Consider now a probability space $(\Omega, \mathfrak{F}, P)$ equipped with a filtration $\mathfrak{F}_t$ (i.e. an increasing family of σ-fields).

Definition 1. A one dimensional Wiener process (or Brownian motion) is usually denoted by $W_t = W_t(\omega)$, $t \geq 0$, $\omega \in \Omega$ or better as $(W_t, \mathfrak{F}_t)$ and is a stochastic process that satisfies the following:

(i). $W_0 = 0$ P-a.s.

(ii). $W_t \backsim N(0, t)$. So W_t is Gaussian.

(iii). W_t has independent increments, i.e. if $0 < t_1 < t_2 < t_3 < t_4$ then $W_{t_4} - W_{t_3}$ is independent of $W_{t_2} - W_{t_1}$.

(iv). It is a continuous process with P-probability one.

Let now $f(t,\omega)$ be a function that is independent of the Wiener process $W.$ after time t. For those functions f that also satisfy the property

$$\int_0^t E|f^2(s,\omega)| < \infty$$

we can define the so called stochastic Itô integral $(If)(t,\omega) = \int_0^t f(s,\omega)dW_s$ which besides the standard properties of an integral, it also satisfies the following two relations:

$$\begin{aligned} E\int_0^t f(s,\omega)dW_s &= 0 \\ E\int_0^t f_1(s,\omega)dW_s \int_0^t f_2(s,\omega)dW_s &= \int_0^t Ef_1f_2(s,\omega)ds \end{aligned}$$

In addition the Itô integral $I_t = (If)(t, \omega)$ is a stochastic process that is an $\mathfrak{F}_t$ martingale, i.e. it satisfies the following properties:

- Let $T > 0$. Then I_t is $\mathfrak{F}_t$ measurable for every $t \in [0, T]$.

- $E|I_t| < \infty$ for every $t \leq T$.

- $I_s = E[I_t|\mathfrak{F}_s]$ for every $s \leq t$ P-a.s.

Let us return now to our parabolic PDE:

$$\begin{aligned} u_t^\epsilon(t,x) &= L^\epsilon u^\epsilon(t,x) \\ u^\epsilon(0,x) &= f(x), \quad (t,x) \in (0,\infty)\times\mathbb{R} \end{aligned} \tag{3}$$

where:

$$L^\epsilon u = \frac{1}{2\epsilon} D'(x/\epsilon) u_x^\epsilon(t,x) + \frac{1}{2} D(x/\epsilon) u_{xx}^\epsilon(t,x) \tag{4}$$

Associated with the elliptic operator L^ϵ is a stochastic process, which we will denote by $X_t^{\epsilon,x} = X_t^{\epsilon,x}(\omega)$ and is the solution to the stochastic differential equation (SDE):

$$X_t^{\epsilon,x} = x + \int_0^t \frac{1}{2\epsilon} D'(X_s^{\epsilon,x}/\epsilon) ds + \int_0^t \sqrt{D(X_s^{\epsilon,x}/\epsilon)} dW_s \tag{5}$$

Taking into account that the Wiener process has independent increments and that equation (5) has a unique (strong) solution, one can show that $X_t^{\epsilon,x}$ has the Markov property. Namely if we define the filtration $\mathfrak{F}_s^{X^{\epsilon,x}} = \sigma(X_v^{\epsilon,x}, v \le s)$, then for every $s \le t$

$$E[X_t^{\epsilon,x} | \mathfrak{F}_s^{X^{\epsilon,x}}] = E[X_t^{\epsilon,x} | X_s^{\epsilon,x}], \ P-\text{a.s.}$$

So we see that there is some connection between elliptic operators of second order (like L^ϵ given in (4)) and It$\hat{o}$ processes (like $X_t^{\epsilon,x}$ that is the solution to (5)). This is a deep result of semigroup theory, but we will not go towards this direction. However this relation will become clearer by the following two famous results:

Theorem 1.[It$\hat{o}$ formula]

Let $g \in \mathfrak{C}^{1,2}((0,\infty) \times \mathbb{R})$ and $X_t^{\epsilon,x}$ be the solution to (5). Then:

$$\begin{aligned} g(t, X_t^{\epsilon,x}) &= g(0,x) + \int_0^t (g_t + L^\epsilon g)(s, X_s^{\epsilon,x}) ds \\ &+ \int_0^t \sqrt{D(X_s^{\epsilon,x}/\epsilon)} g_x(s, X_s^{\epsilon,x}) dW_s \quad (6) \end{aligned}$$

where $L^\epsilon g = \frac{1}{2\epsilon} D'(x/\epsilon) g_x(t,x) + \frac{1}{2} D(x/\epsilon) g_{xx}(t,x)$, is the operator that corresponds to $X_t^{\epsilon,x}$.

Theorem 2.[Feynman-Kac formula]

The solution u^ϵ to (3) can be represented as

$$u^\epsilon(t,x) = Ef(X_t^{\epsilon,x}) \tag{7}$$

Sketch of the proof.

Let t be an arbitrary fixed positive number. Apply It$\hat{o}$ formula to the function $u^\epsilon(t-\hat{t},x)$. By Theorem 1 and after taking $\hat{t}=t$ we have:

$$\begin{aligned} u^\epsilon(0, X_t^{\epsilon,x}) &= u^\epsilon(t,x) + \int_0^t (-u_t^\epsilon + L^\epsilon u^\epsilon)(s, X_s^{\epsilon,x})ds \\ &+ \int_0^t \sqrt{D(X_s^{\epsilon,x}/\epsilon)}u_x^\epsilon(s, X_s^{\epsilon,x})dW_s \end{aligned} \tag{8}$$

By taking expected value now to (8), taking into account the fact that u^ϵ is a solution to (3) and the fact that the stochastic integral is a martingale we conclude that:

$$u^\epsilon(t,x) = Ef(X_t^{\epsilon,x}).$$

Limit as $\epsilon \downarrow 0$ of u^ϵ

The plan is as follows:

(i). Prove that the family $\{X_\cdot^{\epsilon,x}\}$ is weakly compact (tight). In other words this means that there is a stochastic process X_t^x such that for any continuous bounded functional $F(\cdot)$ we have

$$EF(X_\cdot^{\epsilon,x}) \to_{\epsilon\downarrow 0} EF(X_\cdot^x).$$

(ii). Find X_t^x.

(iii). Conclude that $u^\epsilon(t,x) = EF(X_t^{\epsilon,x}) \to_{\epsilon\downarrow 0} EF(X_t^x)$.

(iv). Set $u(t,x) := EF(X_t^x)$ and find the PDE that u satisfies using again Feynman-Kac formula (theorem 2).

Recall now that $X_t^{\epsilon,x}$ satisfies

$$X_t^{\epsilon,x} = x + \int_0^t \frac{1}{2\epsilon} D'(X_s^{\epsilon,x}/\epsilon)ds + \int_0^t \sqrt{D(X_s^{\epsilon,x}/\epsilon)}dW_s \tag{9}$$

At first glance the term $\frac{1}{\epsilon}\int_0^t \frac{1}{2}D'(X_s^{\epsilon,x}/\epsilon)ds$ appears to be of order $\frac{1}{\epsilon}$. However this is **not** true!!!! Indeed:

Let $v(x)$ be a function that we will specify later. Apply It$\hat{o}$ formula to the function $\epsilon v(\epsilon^{-1}x)$ to get:

$$\begin{aligned}\epsilon[v(\epsilon^{-1}X_t^{\epsilon,x}) &- v(\epsilon^{-1}x)] = \int_0^t \epsilon L^\epsilon v(\epsilon^{-1}X_s^{\epsilon,x})ds \\ &+ \int_0^t \sqrt{D(\epsilon^{-1}X_s^{\epsilon,x})}v'(\epsilon^{-1}X_s^{\epsilon,x})dW_s\end{aligned}$$

The last equation and equation (9) motivates us to choose $v(\cdot)$ such that

$$\text{''} \int_0^t \epsilon L^\epsilon v(\epsilon^{-1}X_s^{\epsilon,x})ds = -\frac{1}{\epsilon}\int_0^t \frac{1}{2}D'(X_s^{\epsilon,x}/\epsilon)ds\text{''}.$$

In other words let v satisfy:

$$\frac{1}{2}(D(x)v'(x))' = -\frac{1}{2}D'(x). \tag{10}$$

Equation (10) has many solution. However since D is a periodic function with period one, we want to choose a periodic solution that will also be twice differentiable. Such a choice is attained if we impose the condition $\int_0^1 v(x)dx = 0$. Then the solution to (10) is

$$v(x) = \int_0^x \frac{\overline{D} - D(y)}{D(y)}dy \tag{11}$$

where

$$\overline{D} = (\int_0^1 D^{-1}(y)dy)^{-1}. \tag{12}$$

Thus putting things together we have that (9) can be rewritten as:

$$\begin{aligned} X_t^{\epsilon,x} &= x - \epsilon[v(\epsilon^{-1}X_t^{\epsilon,x}) - v(\epsilon^{-1}x)] \quad (13)\\ &+ \int_0^t \sqrt{D(\epsilon^{-1}X_s^{\epsilon,x})}(1 + v'(\epsilon^{-1}X_s^{\epsilon,x}))dW_s \end{aligned}$$

Therefore we have the following observations:

(i). $X_t^{\epsilon,x}$ is weakly compact. This follows by the boundedness of the coefficients.

(ii). The term $\epsilon[v(\epsilon^{-1}X_t^{\epsilon,x}) - v(\epsilon^{-1}x)]$ in (13) goes to zero as $\epsilon \downarrow 0$ since v is a bounded function. Thus it only remains to consider the stochastic integral

$$\int_0^t \sqrt{D(\epsilon^{-1}X_s^{\epsilon,x})}(1 + v'(\epsilon^{-1}X_s^{\epsilon,x}))dW_s.$$

Lemma [Random time change].

Under technical but fairly general conditions, there is another Wiener process $\tilde{W}_t$ such that for a function f we have

$$\int_0^t f(X_s^{\epsilon,x})dW_s \sim \tilde{W}_{\int_0^t f^2(X_s^{\epsilon,x})ds} := \tilde{W}(\int_0^t f^2(X_s^{\epsilon,x})ds). \tag{14}$$

Thus by the lemma above we have that

$$\begin{aligned} & \int_0^t \sqrt{D(\epsilon^{-1}X_s^{\epsilon,x})}(1+v'(\epsilon^{-1}X_s^{\epsilon,x}))dW_s \quad (15) \\ \sim \quad & \tilde{W}(\int_0^t D(\epsilon^{-1}X_s^{\epsilon,x})(1+v'(\epsilon^{-1}X_s^{\epsilon,x}))^2 ds) \end{aligned}$$

Now we will use the periodicity of $D(x)$. The function $D(x)(1+v'(x))^2$ is 1-periodic, and the Lebesque measure is the invariant measure. Thus a version of the ergodic theorem implies that

$$\begin{aligned} & \int_0^t D(\epsilon^{-1}X_s^{\epsilon,x})(1+v'(\epsilon^{-1}X_s^{\epsilon,x}))^2 ds \\ \to^{\epsilon\downarrow 0} \quad & t\int_0^1 D(x)(1+v'(x))^2 dx \end{aligned} \quad (16)$$

But if we recall the definition of $v(x)$ in (11), we see that

$$\int_0^1 D(x)(1+v'(x))^2 dx = \bar{D} = (\int_0^1 D^{-1}(y)dy)^{-1}. \quad (17)$$

Thus putting things together we see that

$$X_{\cdot}^{\epsilon,x} \rightarrow^{\epsilon\downarrow 0} X_{\cdot}^{x}, \tag{18}$$

where X_t^x satisfies:

$$X_t^x = x + \tilde{W}_{\bar{D}t} = x + \sqrt{\bar{D}} W_t. \tag{19}$$

Therefore the definition of convergence in distribution implies that

$$u^{\epsilon}(t,x) = Ef(X_t^{\epsilon,x}) \rightarrow^{\epsilon\downarrow 0} Ef(X_t^x) = u(t,x) \tag{20}$$

So by Feynman-Kac formula we have that

$$u^{\epsilon}(t,x) \rightarrow^{\epsilon\downarrow 0} u(t,x), \tag{21}$$

where $u(t,x)$ is the solution to:

$$\begin{aligned} u_t(t,x) &= \frac{1}{2}\bar{D}u_{xx}(t,x) \\ u(0,x) &= f(x) \end{aligned} \tag{22}$$

General Remarks

- We actually need to assume something less than periodicity for D.

- Without major changes in the proof we can consider the multi-dimensional case, i.e. when $x \in \mathbb{R}^n$.

- The procedure above can be applied to more complicated problems.

References

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