Reaction Diffusion Equations with Nonlinear Boundary Conditions in Narrow Domains *

Mark Freidlin[†] and Konstantinos Spiliopoulos[‡]
February 2008

[†] *Department of Mathematics, University of Maryland, College Park, 20742, Maryland, USA, E-mail: mif@math.umd.edu*
[‡] *Department of Mathematics, University of Maryland, College Park, 20742, Maryland, USA, kspiliop@math.umd.edu*

Abstract

Second initial boundary problem in narrow domains of width $\epsilon \ll 1$ for linear second order differential equations with nonlinear boundary conditions is considered in this paper. Using probabilistic methods we show that the solution of such a problem converges as $\epsilon \downarrow 0$ to the solution of a standard reaction-diffusion equation in a domain of reduced dimension. This reduction allows to obtain some results concerning wave front propagation in narrow domains. In particular, we describe conditions leading to jumps of the wave front.

Keywords: Reaction Diffusion equations, Narrow Domains, Wave Front Propagation, Instantaneous Reflection

1 Introduction

For each $x \in \mathbb{R}^n$, let D_x be a bounded domain in $\mathbb{R}^m$ with a smooth boundary ∂D_x. Assume, for brevity, that D_x is homeomorphic to a ball in $\mathbb{R}^m$ and contains $0 \in \mathbb{R}^m$. Consider the domain $D = \{(x, y) : x \in \mathbb{R}^n, y \in D_x\} \subset \mathbb{R}^{n+m}$. Assume that the boundary ∂D of D is smooth enough and denote by $\gamma(x, y)$ the inward unit normal to ∂D. Assume that $\gamma(x, y)$ is not parallel to the subspace $\mathbb{R}^n \subset \mathbb{R}^{n+m}$ for any $(x, y) \in \partial D$.

*This paper has been published by Asymptotic Analysis Volume 59, Number 3-4 / 2008, pp.227-249.

Denote by $D^\epsilon, 0<\epsilon<<1$, the domain in $\mathbb{R}^{n+m}$ obtained from D by contraction: $D^\epsilon=\{(x,y): x\in\mathbb{R}^n, y\epsilon^{-1}\in D_x\}$. If $n=1$, D^ϵ is a narrow tube (or a strip for $m=1$) for $0<\epsilon<<1$. If $n>1$, then D^ϵ is a thin layer.

Consider the problem:

$$\begin{aligned}
u^\epsilon_t &= \frac{1}{2}\triangle u^\epsilon, && \text{in } (0,T)\times D^\epsilon \\
u^\epsilon(0,x,y) &= f(x), && \text{on } \{0\}\times D^\epsilon \\
\frac{\partial u^\epsilon}{\partial\gamma^\epsilon} &= -\epsilon c(x,y,u^\epsilon)u^\epsilon, && \text{on } (0,T)\times\partial D^\epsilon,
\end{aligned} \tag{1}$$

where γ^ϵ is the inward unit normal to ∂D^ϵ. The functions f and c are sufficiently regular and bounded; f is assumed to be nonnegative. Our goal in this paper is to study the behavior of solution of problem (1) as $\epsilon\downarrow 0$. Using probabilistic methods, we will prove that $u^\epsilon(t,x,y)$ converges as $\epsilon\downarrow 0$ to the solution $u(t,x)$ of the problem:

$$\begin{aligned}
u_t &= \frac{1}{2}\triangle_x u+\frac{1}{2}\nabla(\log V(x))\nabla_x u+\frac{1}{2}\frac{S(x)}{V(x)}c(x,0,u)u, \quad \text{in } (0,T)\times\mathbb{R}^n \\
u(0,x) &= f(x), \qquad \text{on } \{0\}\times\mathbb{R}^n.
\end{aligned} \tag{2}$$

Here $V(x)$ is the volume of D_x and $S(x)$ is the surface area of ∂D_x. One can expect that, under certain assumptions on the nonlinear term $c(x,y,u)u$ in (1), the solution $u^\epsilon(t,x,y)$ can be approximated by a running-wave-type solution. Corresponding results on the standard reaction diffusion equation (2) (see chapter 6 and 7 in [4]) allow to describe the asymptotic wavefront motion for (1). We will see how the motion of the interface (wavefront) depends on the behavior of the cross-sections D_x of the domain D. In particular, using the results of [4] (chapter 6) we will see that in the case of the nonlinear term of K-P-P type the wavefront can have jumps.

Consider the Wiener process $(X^\epsilon_t, Y^\epsilon_t)$ in D^ϵ with instantaneous normal reflection on the boundary of D^ϵ. Its trajectories can be described by the stochastic differential equations:

$$\begin{aligned}
X^\epsilon_t &= x+W^1_t+\int_0^t \gamma^\epsilon_1(X^\epsilon_s,Y^\epsilon_s)dL^\epsilon_s \\
Y^\epsilon_t &= y+W^2_t+\int_0^t \gamma^\epsilon_2(X^\epsilon_s,Y^\epsilon_s)dL^\epsilon_s.
\end{aligned} \tag{3}$$

Here W^1_t and W^2_t are independent Wiener process in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and (x,y) is a point inside D^ϵ. Moreover γ^ϵ_1 and γ^ϵ_2 are projections of the unit inward normal vector to ∂D^ϵ on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. It is easy to see that

$\lim_{\epsilon\downarrow 0}|\epsilon^{-1}\gamma_1^\epsilon| = \frac{\gamma_1^1}{|\gamma_2^1|}$ and $\lim_{\epsilon\downarrow 0}|\gamma_2^\epsilon| = 1$, where $|\cdot|$ denotes Euclidean length. Furthermore L_t^ϵ is the local time for the process $(X_t^\epsilon, Y_t^\epsilon)$ on ∂D^ϵ, i.e. it is a continuous, non-decreasing process that increases only when $(X_t^\epsilon, Y_t^\epsilon) \in \partial D^\epsilon$ such that the Lebesque measure $\Lambda\{t > 0 : (X_t^\epsilon, Y_t^\epsilon) \in \partial D^\epsilon\} = 0$ (see for instance [11]).

If $(X_t^\epsilon, Y_t^\epsilon)$ is defined by (3), then as it can be derived from Theorem 2.5.1 in [4], $u^\epsilon(t,x,y)$ satisfies the following integral equation in the functional space:

$$u^\epsilon(t,x,y) = E_{x,y} f(X_t^\epsilon) \exp[\int_0^t \epsilon c(X_s^\epsilon, Y_s^\epsilon, u^\epsilon(t-s, X_s^\epsilon, Y_s^\epsilon)) dL_s^\epsilon], \tag{4}$$

where $E_{x,y}$ denotes expectation and the subscript (x,y) denotes the initial point of $(X_s^\epsilon, Y_s^\epsilon)$. Equation (4) has a unique solution if, say, $c(x,y,u)$ has a bounded derivative in u.

Let X_t be the solution of the stochastic differential equation

$$X_t = x + W_t^1 + \int_0^t \frac{1}{2}\nabla(\log V(X_s)) ds. \tag{5}$$

Then the solution $u(t,x)$ of equation (2) satisfies the equality:

$$u(t,x) = E_x f(X_t) \exp[\int_0^t \frac{1}{2}\frac{S(X_s)}{V(X_s)} c(X_s, 0, u(t-s, X_s)) ds]. \tag{6}$$

We prove that the component X_t^ϵ of the process $(X_t^\epsilon, Y_t^\epsilon)$ converges in a certain sense to X_t. This together with uniform in $0 < \epsilon < 1$ bounds for $u^\epsilon(t,x,y)$ and its derivatives allow to prove that the solution of (4) converges to the solution of (6) as $\epsilon \downarrow 0$ uniformly on each compact subset of $[0,\infty) \times \mathbb{R}^{n+m}$.

In the next section we consider averaging of integrals in local time. This result allows in section 3 to prove convergence of the integral in the right side of the first of equations in (3) to the integral term in (5) and convergence of exponents in (4) and (6). Together with a-priori bounds obtained in section 3, this implies convergence of $u^\epsilon(t,x,y)$ to $u(t,x)$. Some results concerning wavefront propagation are presented in section 4.

2 Averaging of Integrals in Local Time

Let $H(x,y)$ be a smooth and bounded function. We want to consider the limiting behavior as $\epsilon \downarrow 0$ of expressions like $\int_0^t \epsilon H(X_s^\epsilon, Y_s^\epsilon/\epsilon) dL_s^\epsilon$ (see Lemma 2.1 below). We will assume that the unit inward normal $\gamma(x,y)$ to ∂D and the function $H(x,y)$ are both three times differentiable in x and y.

Lemma 2.1. Define $Q(x) = \frac{1}{V(x)} \int_{\partial D_x} H(x,y) dS_x$, where dS_x is the surface element on ∂D_x. Then for every $T > 0$ and small enough ϵ, there exists a constant K independent of ϵ such that:

(i). $\sup_{0 \leq t \leq T} E| \int_0^t \frac{1}{2} Q(X_s^\epsilon) ds - \int_0^t \epsilon H(X_s^\epsilon, Y_s^\epsilon/\epsilon) |\gamma_2^\epsilon(X_s^\epsilon, Y_s^\epsilon)| dL_s^\epsilon|^2 \leq K\epsilon^2$.

(ii). For every $\delta > 0$ we have
$P\{\sup_{0 \leq t \leq T} | \int_0^t \frac{1}{2} Q(X_s^\epsilon) ds - \int_0^t \epsilon H(X_s^\epsilon, Y_s^\epsilon/\epsilon) |\gamma_2^\epsilon(X_s^\epsilon, Y_s^\epsilon)| dL_s^\epsilon| > \delta\} \leq K \frac{\epsilon^2}{\delta^2}$.

The proof of lemma 2.1 relies on the following lemma, which we prove first.

Lemma 2.2. For every $T > 0$ and small enough ϵ, there exists a constant K_1 independent of ϵ such that:

$$E|\epsilon L_T^\epsilon|^2 \leq K_1$$

Proof.
Consider the auxiliary problem

$$\begin{aligned} \triangle_y v(x,y) &= Q(x), \quad y \in D_x \subset \mathbb{R}^m \\ \frac{\partial_y v(x,y)}{\partial n(x,y)} &= -1, \quad y \in \partial D_x, \end{aligned} \tag{7}$$

where $n(x,y) = \frac{\gamma_2^1(x,y)}{|\gamma_2^1(x,y)|}$ and $x \in \mathbb{R}^n$ is a parameter. Let

$$Q(x) = \frac{S(x)}{V(x)}, \tag{8}$$

where $S(x)$ is the surface area of D_x and $V(x)$ is the volume of D_x. As it can be derived from [1], a smooth in x and y solution $v(x,y)$ of problem (7) exists and is bounded together with its first and second derivatives. So we can apply Itô formula to the function $\epsilon v(x, y/\epsilon)$, and get:

$$\begin{aligned} \epsilon^2 v(X_t^\epsilon, Y_t^\epsilon/\epsilon) &= \epsilon^2 v(x, y/\epsilon) + \int_0^t \epsilon^2 \frac{1}{2} \triangle_x v(X_s^\epsilon, Y_s^\epsilon/\epsilon) ds + \int_0^t \frac{1}{2} \triangle_y v(X_s^\epsilon, Y_s^\epsilon/\epsilon) ds \\ &+ \int_0^t \epsilon^2 (\nabla_x v(X_s^\epsilon, Y_s^\epsilon/\epsilon), dW_s^1) + \int_0^t \epsilon (\nabla_y v(X_s^\epsilon, Y_s^\epsilon/\epsilon), dW_s^2) \\ &+ \int_0^t \epsilon^2 (\nabla_x v(X_s^\epsilon, Y_s^\epsilon/\epsilon), \gamma_1^\epsilon(X_s^\epsilon, Y_s^\epsilon)) dL_s^\epsilon \\ &+ \int_0^t \epsilon (\nabla_y v(X_s^\epsilon, Y_s^\epsilon/\epsilon), \gamma_2^\epsilon(X_s^\epsilon, Y_s^\epsilon)) dL_s^\epsilon \end{aligned} \tag{9}$$

Recalling now that $\lim_{\epsilon\downarrow 0}|\epsilon^{-1}\gamma_1^\epsilon| = \frac{\gamma_1^1}{|\gamma_2^1|}$ and $\lim_{\epsilon\downarrow 0}|\gamma_2^\epsilon| = 1$ and that v satisfies (7) one easily concludes that there is an $\epsilon_0 = \epsilon_0(\||\nabla_x v|\|, \gamma_1^1) > 0$ such that for every $\epsilon < \epsilon_0$:

$$\begin{aligned} E|\epsilon L_T^\epsilon|^2 &\leq C[\epsilon^4(2\|v^2\| + \|\frac{1}{2}\triangle_x v\|^2 T^2 + \||\nabla_x v|^2\| T) + \\ &+ \epsilon^2 \||\nabla_y v|^2\| T + \|\frac{1}{2} Q\|^2 T], \end{aligned}$$

where for any function g, $\|g\| = \sup_z |g(z)|$. Here, we also used the fact that the local time is increasing function of t. This proves the statement of the lemma.

□

Proof of Lemma 2.1. We consider the auxiliary problem

$$\begin{aligned} \triangle_y v(x,y) &= Q(x), \quad y \in D_x \subset \mathbb{R}^m \\ \frac{\partial_y v(x,y)}{\partial n(x,y)} &= -H(x,y), \quad y \in \partial D_x, \end{aligned} \tag{10}$$

where $n(x,y) = \frac{\gamma_2^1(x,y)}{|\gamma_2^1(x,y)|}$ and $x \in \mathbb{R}^n$ is a parameter.

The necessary and sufficient condition for the existence of a solution for (10) is that

$$Q(x) = \frac{1}{V(x)} \int_{\partial D_x} H(x,y) dS_x, \tag{11}$$

where dS_x is the surface element on ∂D_x and $V(x) = \text{volume}(D_x)$.

Applying Itô formula to the function $\epsilon v(x, y/\epsilon)$ and using the bounds obtained in Lemma 2.2 we get the following inequalities:

$$\begin{aligned} &\sup_{0\leq t\leq T} E|\int_0^t \frac{1}{2} Q(X_s^\epsilon) ds - \int_0^t \epsilon H(X_s^\epsilon, Y_s^\epsilon/\epsilon)|\gamma_2^\epsilon(X_s^\epsilon, Y_s^\epsilon)| dL_s^\epsilon|^2 \leq \\ \leq \; &\epsilon^4 C(2\|v^2\| + \|\frac{1}{2}\triangle_x v\|^2 T^2 + \||\nabla_x v|^2\| T + \||\nabla_x v|\|^2 K_1) + \epsilon^2 C \||\nabla_y v|^2\| T, \end{aligned}$$

which proves statement (i) of the lemma.

For part (ii) one makes use of the Doob maximal inequalities (see [11], page 14):

$$\begin{aligned} E[\sup_{0\leq t\leq T} |\int_0^t (\nabla_x v(X_s^\epsilon, Y_s^\epsilon/\epsilon), dW_s^1)|]^2 &\leq 4 \parallel |\nabla_x v|^2 \parallel T \\ E[\sup_{0\leq t\leq T} |\int_0^t (\nabla_y v(X_s^\epsilon, Y_s^\epsilon/\epsilon), dW_s^2)|]^2 &\leq 4 \parallel |\nabla_y v|^2 \parallel T \end{aligned}$$

Then, following the procedure that proved part (i) we get that there is an $\epsilon_0 > 0$ such that for every $\epsilon < \epsilon_0$:

$$E[\sup_{0\leq t\leq T}|\int_0^t \frac{1}{2}Q(X^\epsilon_s)ds - \int_0^t \epsilon H(X^\epsilon_s, Y^\epsilon_s/\epsilon)|\gamma^\epsilon_2(X^\epsilon_s, Y^\epsilon_s)|dL^\epsilon_s|]^2 \leq$$
$$\leq \quad \epsilon^4 C(4\|v^2\| + \|\frac{1}{2}\triangle_x v\|^2 T^2 + 4\||\nabla_x v|^2\|T + \||\nabla_x v|\|^2 K_1) + \epsilon^2 C\||\nabla_y v|^2\|T,$$

which together with Chebyshev inequality proves statement (ii) of the lemma.

□

3 Limit of u^ϵ.

In this section we consider the limit as $\epsilon \to 0$ of the solution u^ϵ to problem (1). The result is given in Theorem 3.4. The proof will proceed as follows. First (in Proposition 3.2.) we write down an integral equation in the space of trajectories for the solution of (1). Then in Lemma 3.3 we consider the mean square limit as $\epsilon \to 0$ of the underlying stochastic process with instantaneous normal reflection on the boundary of D^ϵ (see (3)). Lastly an important ingredient to the proof are the a-priori bounds for u^ϵ and its derivatives. These a-priori bounds are independent of ϵ, their derivation is standard and are given for completeness in Proposition 3.7.

We assume that the initial function $f(x)$ of problem (1) is bounded, non-negative and can have finite number of simple discontinuities. The function $c(x, y, u)$ is assumed to be uniformly bounded in all arguments, continuous in x,y, Lipschitz continuous in u and that there exist constants $M, N > 0$ such that $c(\cdot, \cdot, u) < -M$ for $u > N$.

In addition we assume that the boundary of D^1 satisfies $\partial D^1 \in \mathcal{C}^{3+a}(\mathbb{R}^m)$, where $a \in (0, 1)$.

Remark 3.1. For the existence of a classical solution to (1) one actually needs only to assume $\partial D^1 \in \mathcal{C}^{2+a}(\mathbb{R}^m)$. The assumption $\partial D^1 \in \mathcal{C}^{3+a}(\mathbb{R}^m)$ is being done solely for the purpose of Lemma 3.2 and Theorem 3.3.

Let $(X^\epsilon, Y^\epsilon, L^\epsilon)$ in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1_+$ satisfy (3). Then we have:

Proposition 3.2. Problem (1) has a unique classical solution in $[0, T) \times D^\epsilon$ which satisfies:

$$u^\epsilon(t, x, y) = E_{x,y} f(X^\epsilon_t) \exp[\int_0^t \epsilon c(X^\epsilon_s, Y^\epsilon_s, u^\epsilon(t-s, X^\epsilon_s, Y^\epsilon_s))dL^\epsilon_s]. \qquad (12)$$

Proof. Under our assumptions, the uniqueness and existence of a classical solution to (1) follows from Theorem 7.5.13 of [9]). The equation (12) follows from Theorem 2.5.1 of [4].

□

In order now to consider the limit as $\epsilon \to 0$ of (12), we need first to examine the asymptotic behavior of X_t^ϵ as $\epsilon \to 0$.

We will prove that X_t^ϵ converges as $\epsilon \downarrow 0$ to X_t, where X_t is the solution to

$$X_t = x + W_t^1 + \int_0^t \frac{1}{2}\nabla(\log V(X_s))ds, \tag{13}$$

where $V(x) = \text{volume}(D_x)$. Hence, we see that as $\epsilon \downarrow 0$, the effect of the reflection on the boundary is an extra drift term. A sketch of the proof for the above result is given in chapter 7 of [6]. More details are given here.

Lemma 3.3. For any $T > 0$ we have

$$\sup_{0\leq t\leq T} E_x|X_t^\epsilon - X_t|^2 \to 0 \text{ as } \epsilon \to 0. \tag{14}$$

Proof. It is not difficult to see that $\gamma_1^\epsilon(x,y) = \epsilon\frac{\gamma_1^1(x,y)}{|\gamma_2^1(x,y)|}|\gamma_2^\epsilon(x,y)|$ and that

$$\int_{\partial D_x} \frac{\gamma_1^1(x,y)}{|\gamma_2^1(x,y)|}dS_x = \nabla V(x). \tag{15}$$

Then Lemma 2.1 with $H(x,y) = \frac{\gamma_1^1(x,y)}{|\gamma_2^1(x,y)|}$ and $Q(x) = \nabla \log V(x)$ implies that for small enough ϵ there exists a constant K independent of ϵ such that

$$\sup_{0\leq t\leq T} E|\int_0^t \frac{1}{2}\nabla\log(V(X_s^\epsilon))ds - \int_0^t \gamma_1^\epsilon(X_s^\epsilon, Y_s^\epsilon)dL_s^\epsilon|^2 \leq \epsilon^2 K. \tag{16}$$

Now we write

$$\begin{aligned}
X_t^\epsilon - X_t &= \int_0^t \gamma_1^\epsilon(X_s^\epsilon, Y_s^\epsilon)dL_s^\epsilon - \int_0^t \frac{1}{2}\nabla\log(V(X_s))ds \\
&= [\int_0^t \gamma_1^\epsilon(X_s^\epsilon, Y_s^\epsilon)dL_s^\epsilon - \int_0^t \frac{1}{2}\nabla\log(V(X_s^\epsilon))ds] \\
&+ [\int_0^t \frac{1}{2}\nabla\log(V(X_s^\epsilon))ds - \int_0^t \frac{1}{2}\nabla\log(V(X_s))ds]
\end{aligned} \tag{17}$$

Then Gronwall Lemma and (16) give:

$$\sup_{0\leq t\leq T} E_x|X_t^\epsilon - X_t|^2 \to 0 \text{ as } \epsilon \to 0, \tag{18}$$

which is the statement of the lemma.

□

Consider now the solution, u, to the equation

$$u(t,x) = E_x f(X_t) \exp[\int_0^t \overline{c}(X_s, u(t-s, X_s))ds], \tag{19}$$

where

$$\overline{c}(x, u(t,x)) = \frac{1}{2}\frac{S(x)}{V(x)} c(x, 0, u(t,x)). \tag{20}$$

For notational convenience we will also denote $\overline{c}(t,x) = \overline{c}(x, u(t,x))$.

Since $\overline{c}(x,u)$ is Lipschitz continuous in u, the solution of (19) exists and is unique. Our assumptions on the functions f, c and the boundary ∂D_x, imply that the solution u to (19) is actually the classical solution of the following parabolic problem:

$$\begin{aligned} u_t &= \frac{1}{2}\triangle_x u + \frac{1}{2}\nabla(\log V(x)) \cdot \nabla_x u + \frac{1}{2}\frac{S(x)}{V(x)} c(x, 0, u(t,x))u, \text{ in } (0,T) \times \mathbb{R}^n \\ u(0,x) &= f(x), \text{ on } \{0\} \times \mathbb{R}^n. \end{aligned} \tag{21}$$

Theorem 3.4. Under our assumptions, we have that

$u^\epsilon(t,x,y) \to u(t,x)$ as $\epsilon \to 0$, uniformly in any compact sunset of $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m$,

where $u^\epsilon(t,x,y)$, $u(t,x)$ are the solutions to (1) and (21) respectively.

Proof. By Proposition 3.6 and the well known theorem of Ascoli-Arzela we get that there exists a subsequence of $\{u^\epsilon\}$ (which for convenience we will denote again by $\{u^\epsilon\}$) and a function u, such that:

$$u^\epsilon \to u \text{ as } \epsilon \to 0, \text{ uniformly in compacts.}$$

We will prove that u actually satisfies (19) which then implies that u satisfies (21). Fix t and x and consider the solution $v(y) = v^{\epsilon,t,x}(y)$ to the elliptic boundary value problem:

$$\begin{aligned} \triangle_y v(y) &= \overline{c}^\epsilon(t,x), \quad y \in D_x \subset \mathbb{R}^m \\ \frac{\partial_y v(y)}{\partial n(x,y)} &= -\frac{1}{|\gamma_2^\epsilon(x, \epsilon y)|} c(x, \epsilon y, u^\epsilon(t, x, \epsilon y)), \quad y \in \partial D_x. \end{aligned} \tag{22}$$

Problem (22) is solvable if

$$\overline{c}^\epsilon(t,x) = \frac{1}{V(x)} \int_{\partial D_x} \frac{1}{|\gamma_2^\epsilon(x, \epsilon y)|} c(x, \epsilon y, u^\epsilon(t, x, \epsilon y)) dS_x. \tag{23}$$

Proceeding similarly now to Lemma 2.1 and recalling that v satisfies (22), we see that there is a constant $K^{\epsilon} = K(\|\nabla_x v^{\epsilon}\|, \|\nabla_y v^{\epsilon}\|, \|\triangle_x v^{\epsilon}\|, \|\triangle_y v^{\epsilon}\|, \|v^{\epsilon}_t\|, \|\gamma^1_1\|, T)$ such that:

$$\begin{aligned} &\sup_{0\leq t\leq T} E|\int_0^t \frac{1}{2}\overline{c}^{\epsilon}(t-s, X^{\epsilon}_s)ds - \int_0^t \epsilon c(X^{\epsilon}_s, Y^{\epsilon}_s, u^{\epsilon}(t-s, X^{\epsilon}_s, Y^{\epsilon}_s))dL^{\epsilon}_s|^2 \\ \leq\ & \epsilon^2 K^{\epsilon}(1+\sup_{0\leq t\leq T} E[\epsilon L^{\epsilon}_t]^2) \end{aligned} \tag{24}$$

We observe that K^{ϵ} depends on ϵ only through functions that are uniformly bounded in ϵ (Proposition 3.7). This observation and Lemma 2.2 imply that as $\epsilon \longrightarrow 0$:

$$\sup_{0\leq t\leq T} E|\int_0^t \frac{1}{2}\overline{c}^{\epsilon}(t-s, X^{\epsilon}_s)ds - \int_0^t \epsilon c(X^{\epsilon}_s, Y^{\epsilon}_s, u^{\epsilon}(t-s, X^{\epsilon}_s, Y^{\epsilon}_s))dL^{\epsilon}_s|^2 \rightarrow 0. \tag{25}$$

Moreover the Lebesque dominated convergence Theorem, Lemma 3.3, the compactness of the family $\{u^{\epsilon}\}$ and (23), imply that as $\epsilon \rightarrow 0$:

$$\sup_{0\leq t\leq T} E|\int_0^t \frac{1}{2}\overline{c}^{\epsilon}(t-s, X^{\epsilon}_s)ds - \int_0^t \overline{c}(t-s, X_s)ds|^2 \rightarrow 0, \tag{26}$$

where $\overline{c}^{\epsilon}, \overline{c}$ and X_t are given by (23), (20) and (13) respectively.

Now let $u^{\epsilon}(t,x,y), u(t,x)$ be the solutions to (12) and (19) respectively. Taking into account relations (25), (26), the weak convergence of X^{ϵ}_t to X_t as $\epsilon \rightarrow 0$ (which is implied by Lemma 3.3) and Proposition 3.2 we get the statement of the Theorem.

□

We conclude this section with the a-priori bounds for the Hölder norm of the solution and for the sup-norm of the solution, the first and the second derivatives of the solution of (1). These bounds will be uniform in ϵ. The method follows closely [9].

Let us first introduce some notation.

We write $U^{\epsilon}_T = [0,T) \times D^{\epsilon}$, $\overline{U}^{\epsilon}_T = [0,T) \times \overline{D}^{\epsilon}$, $\partial U^{\epsilon}_T = [0,T) \times \partial D^{\epsilon}$ and $V^{\epsilon}_T = (0,T) \times D^{\epsilon}$, where $\overline{D}^{\epsilon} = D^{\epsilon} \cup \partial D^{\epsilon}$.

For $0 < a < 1$, $T > 0$ and for any function g we write:

$$\begin{aligned} \| g \|_{U^{\epsilon}_T} &= \sup_{(t,z)\in U^{\epsilon}_T} |g(t,z)| \\ \|H^a g\|_{U^{\epsilon}_T} &= \sup_{(t,z),(t',z')\in U^{\epsilon}_T} \frac{|g(t,z)-g(t',z')|}{|t-t'|^{a/2}+|z-z'|^a} \end{aligned}$$

$$\begin{aligned} \|g\|_{U^{\epsilon}_{T},a} &= \|g\|_{U^{\epsilon}_{T}} + \|H^{a}g\|_{U^{\epsilon}_{T}} \\ \|g\|_{D^{\epsilon},T,1+a} &= \|g\|_{U^{\epsilon}_{T},a} + \|g_{t}\|_{U^{\epsilon}_{T}} + \|Dg\|_{(0,T)\times D^{\epsilon}} \\ \overline{\|g\|}_{D^{\epsilon},T,1+a} &= \|g\|_{\overline{D}^{\epsilon},T,1+a} \end{aligned}$$

Moreover for notational convenience we will write $z=(x,y)$.

Lemma 3.5. Under our assumptions there exists a constant C_1, independent of $\epsilon>0$, such that

$$0 \leq u^{\epsilon} \leq C_1 \text{ in } \overline{U}^{\epsilon}_{T}.$$

Proof. Lemma 3.5 can be proven using equation (12). Here we give an analytic proof of the claim. For any fixed $b>0$ we define the function

$$w^{\epsilon} = (u^{\epsilon}-b)^{+} = \max\{u^{\epsilon}-b, 0\}.$$

It is easy to show that

$$w^{\epsilon}_{t} \leq \frac{1}{2}\triangle w^{\epsilon} \qquad \text{on } (0,T)\times D^{\epsilon}.$$

in the weak sense. Let us choose now $b=\max\{N, \|f\|\}$, where N is such that if $u>N$ then $c(\cdot,\cdot,u)<-M$ for some $M>0$. Then

$$w^{\epsilon}(0,x,y)=0.$$

Let us now assume that w^{ϵ} attains a maximum positive value on the boundary $\partial V^{\epsilon}_{T}$ at the point (t_o,x_o,y_o). Since u^{ϵ} is continuous up to the boundary, there exists a connected set Δ such that $(t_o,x_o,y_o)\in\Delta$, $\Delta\subset\partial V^{\epsilon}_{T}$ and $w^{\epsilon}>0$ on Δ, i.e. $u^{\epsilon}>b$ on Δ. Since (t_o,x_o,y_o) is a maximum for w^{ϵ} and γ^{ϵ} is the inward normal derivative we get that $\frac{\partial w^{\epsilon}}{\partial\gamma^{\epsilon}}\leq 0$ at (t_o,x_o,y_o). But on Δ we have that $\frac{\partial w^{\epsilon}}{\partial\gamma^{\epsilon}} = \frac{\partial u^{\epsilon}}{\partial\gamma^{\epsilon}} = -\epsilon c(x,y,u^{\epsilon})u^{\epsilon}$. Taking into account the particular choice of b and that $c(\cdot,\cdot,u)<-M$ for $u>N$, we get that $-\epsilon c(x,y,u^{\epsilon})u^{\epsilon}>0$ at (t_o,x_o,y_o). Thus we have a contradiction and so maximum principle implies that

$$w^{\epsilon}=0 \Longrightarrow u^{\epsilon}<b \text{ in } \overline{U}^{\epsilon}_{T}.$$

Lastly maximum principle again implies that $u^{\epsilon}\geq 0$.

□

Let us consider the following linear parabolic pde:

$$\begin{aligned} v^{\epsilon}_{t} &= \frac{1}{2}\triangle v^{\epsilon}, && \text{in } (0,T)\times D^{\epsilon} \\ v^{\epsilon}(0,x,y) &= f(x), && \text{on } \{0\}\times D^{\epsilon} \\ \frac{\partial v^{\epsilon}}{\partial\gamma^{\epsilon}} &= -\epsilon c(x,y)v^{\epsilon}, && \text{on } (0,T)\times\partial D^{\epsilon}, \end{aligned} \qquad (27)$$

where f, c are bounded smooth functions. Under the standard hypotheses problem (27) has a unique classical solution (Theorem 5.3.2 in [9]).

Lemma 3.6. There is a constant C, independent of ϵ, and an open set $I \subset (0,1)$ such that for any $a \in I$:

$$\overline{\|v^{\epsilon}\|}_{D^{\epsilon},T,1+a} + \|D^{2}v^{\epsilon}\|_{V_{T}^{\epsilon}} \leq C. \tag{28}$$

Proof. We will give just a sketch of the proof, since the analysis follows [14], [15], [16] and [9]. The calculations are lengthy but standard.

We solve the second initial-boundary value problem (27) by reducing it to an integral equation, i.e. we write:

$$v^{\epsilon}(t,z) = \int_{0}^{t}\int_{\partial D^{\epsilon}} \Gamma^{\epsilon}(t,z,\tau,\xi)\phi^{\epsilon}(\tau,\xi)d\partial D^{\epsilon}_{\xi}d\tau + \int_{D^{\epsilon}} \Gamma^{\epsilon}(t,z,0,\xi)f(\xi)d\xi, \tag{29}$$

where $\Gamma^{\epsilon}(t,z,\tau,\xi) = (2\sqrt{\pi})^{-n-m}(t-\tau)^{-\frac{n+m}{2}}\exp[-\frac{\sum_{i=1}^{n+m}(z_{i}-\xi_{i})^{2}}{4(t-\tau)}]$ is the fundamental solution to the heat equation and $\phi(t,z)$ is the solution to a Voltera type integral equation:

$$\begin{aligned} \phi^{\epsilon}(t,z) &= 2\int_{0}^{t}\int_{\partial D^{\epsilon}}[\frac{\partial \Gamma^{\epsilon}(t,z,\tau,\xi)}{\partial \gamma^{\epsilon}} + \epsilon c(z)\Gamma^{\epsilon}(t,z,\tau,\xi)]\phi^{\epsilon}(\tau,\xi)d\partial D^{\epsilon}_{\xi}d\tau \\ &+ 2[\int_{D^{\epsilon}} \frac{\partial \Gamma^{\epsilon}(t,z,0,\xi)}{\partial \gamma^{\epsilon}} f(\xi)d\xi + \epsilon c(z)\int_{D^{\epsilon}}\Gamma^{\epsilon}(t,z,0,\xi)f(\xi)d\xi] \end{aligned} \tag{30}$$

Let us now define

$$\begin{aligned} F^{\epsilon}(t,z) &= \int_{D^{\epsilon}} \frac{\partial \Gamma^{\epsilon}(t,z,0,\xi)}{\partial \gamma^{\epsilon}} f(\xi)d\xi + \epsilon c(z)\int_{D^{\epsilon}}\Gamma^{\epsilon}(t,z,0,\xi)f(\xi)d\xi \\ M_{1}(t,z,\tau,\xi) &= \frac{\partial \Gamma^{\epsilon}(t,z,\tau,\xi)}{\partial \gamma^{\epsilon}} + \epsilon c(z)\Gamma^{\epsilon}(t,z,\tau,\xi) \\ M_{\nu+1}(t,z,\tau,\xi) &= \int_{0}^{t}\int_{\partial D^{\epsilon}} M_{1}(t,z,t',z')M_{\nu}(t',z',\tau,\xi)d\partial D^{\epsilon}_{z'}dt' \end{aligned}$$

It can be shown (see [9]) that there is a Hölder continuous (in space variables) and bounded (with bound and Hölder coefficient independent of ϵ) solution ϕ^{ϵ} for (30), expressed in the form:

$$\phi^{\epsilon}(t,x) = 2F^{\epsilon}(t,z) + 2\sum_{\nu=1}^{\infty}\int_{0}^{t}\int_{\partial D^{\epsilon}} M_{\nu}(t,z,\tau,\xi)F^{\epsilon}(\tau,\xi)d\partial D^{\epsilon}_{\xi}d\tau \tag{31}$$

Using the boundedness and the Hölder continuity of (31) and (29), one can show (see [14], [15], [16] and [9]) that there is a constant C, independent of ϵ, such that

$$\overline{\|v^{\epsilon}\|}_{D^{\epsilon},T,1+a} + \|D^{2}v^{\epsilon}\|_{V_{T}^{\epsilon}} \leq C.$$

□

Now we are ready to prove the result for the a-priori bounds:

Proposition 3.7. There is a constant C, independent of ϵ, and an open set $I \subset (0,1)$ such that for any $b > a \in I$ (a is the constant from Lemma 3.6.):

$$\overline{\|u^{\epsilon}\|}_{D^{\epsilon},T,1+b} + \|D^{2}u^{\epsilon}\|_{V_{T}^{\epsilon}} \leq C. \tag{32}$$

where u^{ϵ} is a classical solution to (1).

Proof. We will use Schauder's fixed point Theorem. Let us first define for convenience $\overline{\|\cdot\|}_{2+a} = \overline{\|\cdot\|}_{D^{\epsilon},T,1+a} + \|D^{2}\cdot\|_{V_{T}^{\epsilon}}$.

Let $\mathcal{C}^{2+a}$ be the Banach space of all functions $u^{\epsilon}(t,z)$ that are continuous in $\overline{U}_{T}^{\epsilon}$ with norm $\overline{\|u^{\epsilon}\|}_{2+a}$.

For any $C>0$, let $\mathcal{C}_{C}^{2+a}$ be the set $\{u^{\epsilon} : u^{\epsilon} \in \mathcal{C}^{2+a}, \overline{\|u^{\epsilon}\|}_{2+a} \leq C\}$.

For every $u^{\epsilon} \in \mathcal{C}_{C}^{2+a}$ define $w^{\epsilon} = Tu^{\epsilon}$ to be the solution to the following problem:

$$\begin{aligned} w_{t}^{\epsilon} &= \frac{1}{2}\triangle w^{\epsilon}, && \text{in } (0,T)\times D^{\epsilon} \\ w^{\epsilon}(0,z) &= f(x), && \text{on } \{0\}\times D^{\epsilon} \\ \frac{\partial w^{\epsilon}}{\partial \gamma^{\epsilon}} &= -\epsilon c(z,u^{\epsilon})w^{\epsilon}, && \text{on } (0,T)\times \partial D^{\epsilon}, \end{aligned} \tag{33}$$

Then similarly as in Lemma 3.6, one can write:

$$w^{\epsilon}(t,z) = \int_{0}^{t}\int_{\partial D^{\epsilon}} \Gamma^{\epsilon}(t,z,\tau,\xi)\phi^{\epsilon}(\tau,\xi)d\partial D^{\epsilon}_{\xi}d\tau + \int_{D^{\epsilon}}\Gamma^{\epsilon}(t,z,0,\xi)f(\xi)d\xi, \tag{34}$$

where $\phi^{\epsilon}(t,z)$ satisfies:

$$\begin{aligned} \phi^{\epsilon}(t,z) &= 2\int_{0}^{t}\int_{\partial D^{\epsilon}}[\frac{\partial\Gamma^{\epsilon}(t,z,\tau,\xi)}{\partial\gamma^{\epsilon}} + \epsilon c(z,u^{\epsilon})\Gamma^{\epsilon}(t,z,\tau,\xi)]\phi^{\epsilon}(\tau,\xi)d\partial D^{\epsilon}_{\xi}d\tau \\ &+ 2[\int_{D^{\epsilon}}\frac{\partial\Gamma^{\epsilon}(t,z,0,\xi)}{\partial\gamma^{\epsilon}}f(\xi)d\xi + \epsilon c(z,u^{\epsilon})\int_{D^{\epsilon}}\Gamma^{\epsilon}(t,z,0,\xi)f(\xi)d\xi] \end{aligned} \tag{35}$$

We shall prove that T has a fixed point.

Since u^{ϵ} and c are bounded functions, one can show, in the same way as in the proof of Lemma 3.6, that the function $\phi^{\epsilon}(t,z)$ that satisfies (35) is bounded and Hölder continuous (in space variables) with bound and Hölder constant independent of ϵ.

Using this result and representation (34) one can conclude (Lemma 3.6) that there is a constant C such that

$$\overline{\|w^{\epsilon}\|}_{2+a} \leq C.$$

So T maps $\mathcal{C}^{2+a}_{C}$ into itself for an appropriately chosen constant C.

Now let $\{u^{\epsilon}_{n}\}$ be a sequence of functions that belong to $\mathcal{C}^{2+a}_{C}$ and $w^{\epsilon}_{n}, \phi^{\epsilon}_{n}$ be defined by (34) and (35) when $u^{\epsilon} = u^{\epsilon}_{n}$. Assume that $\overline{\|u^{\epsilon}_{n} - u^{\epsilon}\|}_{2+a} \to 0$ as $n \to \infty$. We need to show that $\overline{\|w^{\epsilon}_{n} - w^{\epsilon}\|}_{2+a} \to 0$ as $n \to \infty$.

The continuity of the function $c(z, u)$ in u-variables imply that $\|\phi^{\epsilon}_{n} - \phi^{\epsilon}\|_{U^{\epsilon}_{T}} \to 0$ as $n \to \infty$. This and (34) give us $\overline{\|w^{\epsilon}_{n} - w^{\epsilon}\|}_{2+a} \to 0$.

Therefore T is a continuous map.

Next we need to show that T maps $\mathcal{C}^{2+a}_{C}$ into a compact subset of $\mathcal{C}^{2+a}_{C}$. This is an easy consequence of Theorem 7.1.1 of [9], which states that for $0 < a < b < 1$, the bounded subsets of $\mathcal{C}^{2+b}$ are pre-compact subsets of $\mathcal{C}^{2+a}$.

Lastly $\mathcal{C}^{2+b}_{C}$ is a closed convex set of the Banach space $\mathcal{C}^{2+b}$.

Therefore by Schauder's Fixed Point Theorem we get that T has a fixed point, i.e. there exists a u^{ϵ} such that $u^{\epsilon} = Tu^{\epsilon}$ and actually

$$u^{\epsilon} = Tu^{\epsilon} \in \mathcal{C}^{2+b}_{C}.$$

□

4 Some Results On Wave Front Propagation

In this section we will see some applications of Theorem 3.4 to the question of wave front propagation in narrow domains. As we mentioned in the introduction, corresponding results on the standard reaction diffusion equation (2) (see chapter 6 and 7 in [4], [10] and [13]) allow to describe the asymptotic wavefront motion for (1).

We will focus on two different cases. In subsection 4.1 we consider the case where the functions $c(\cdot, u)$, $V(\cdot)$, $S(\cdot)$ and $f(\cdot)$ change slowly in x, i.e. $c(\cdot, u) = c(\delta x, u)$, $V(\cdot) = V(\delta x)$, $S(\cdot) = S(\delta x)$ and $f(\cdot) = f(\delta x)$ for $0 < \delta \ll 1$. We first assume that the nonlinear boundary term in (1), $c(x, y, u)$, is of K-P-P type for $y = 0$, i.e. $c(x, 0, u)$ is positive for $u < 1$, negative for $u > 1$ and $c(x) = c(x, 0, 0) = \max_{0 \leq u \leq 1} c(x, 0, u)$. We will see how the motion of the wavefront depends on the behavior of the cross-sections D_x of the domain D. In particular, using the results of [4] (chapter 6) we will see that in the case of the nonlinear term of K-P-P type and for $x \in \mathbb{R}$ the wavefront can have jumps. Actually, the jumps of the wavefront appear at positions where the tube becomes thinner. The results are given in Theorem 4.1.2, Theorem 4.1.5 and Theorem 4.1.7. Then we briefly discuss the bistable case, i.e. when $c(x,0,u) > 0$ for $u \in (\mu, 1)$ and $c(x,0,u) < 0$ for $u \in (0,\mu) \cup (1,\infty)$, where $0 < \mu < 1$. In this case we consider a specific example and we will see how the asymptotic speed of the wavefront depends on the surface area to volume ratio $\frac{S(x)}{V(x)}$. In subsection 4.2, we return to the K-P-P case, but now we consider front propagation when $x \in \mathbb{R}$ and the boundary ∂D^1 of D^1 is determined by stationary random processes on $\mathbb{R}$ on some probability space $(\hat{\Omega}, \hat{\mathfrak{F}}, \hat{P})$. The conclusion is in Theorem 4.2.7.

We will denote by $\overline{c}(x,u) := \frac{1}{2}\frac{S(x)}{V(x)}c(x,0,u(t,x))$ the nonlinear term in (2). Obviously the type of $\overline{c}(x,u)$ (K-P-P or bistable) is determined by $c(x,0,u)$ and vice-versa.

4.1 Wave Fronts in Slowly Changing Media

Let us assume that the functions $c(\cdot,u)$, $V(\cdot)$, $S(\cdot)$ and $f(\cdot)$ change slowly in x, i.e. $c(\cdot,u) = c(\delta x, u)$, $V(\cdot) = V(\delta x)$, $S(\cdot) = S(\delta x)$ and $f(\cdot) = f(\delta x)$ for $0 < \delta \ll 1$.

We start with the case where the nonlinear term $\overline{c}(x,u)$ of (2) is of K-P-P type. We additionally assume that the closure of the support of f, F_o, coincides with the closure of its interior. Lastly we take for brevity $x \in \mathbb{R}^1$ and $\overline{c}(x) = \overline{c}(x,0) = \frac{1}{2}\frac{S(x)}{V(x)}c(x,0,0)$ (recall that $c(x,0,0) = \sup_{0\le u\le 1} c(x,0,u)$) to be an increasing function.

Let $\phi : [0,T] \to \mathbb{R}^1$ and introduce the functional

$$R_{0,T}(\phi) = \begin{cases} \int_0^T [\overline{c}(\phi_s) - \frac{1}{2}|\dot{\phi}_s|^2] ds, & \phi \text{ is absolutely continuous} \\ +\infty, & \text{for the rest of } \mathcal{C}_{0,T}. \end{cases} \tag{36}$$

Put

$$W(t,x) = \sup\{R_{0,t}(\phi) : \phi \in \mathcal{C}_{0,t}(\mathbb{R}^1), \phi_0 = x, \phi_t \in F_o\}. \tag{37}$$

We say that condition (N) is satisfied if for any $t > 0$ and $(t,x) \in \{(t,x) : W(t,x) = 0\}$:

$$W(t,x) = \sup\{R_{0,t}(\phi) : \phi_0 = x, \phi_t \in F_o, (t-s, \phi_s) \in \{(t,x) : W(t,x) < 0\}\}.$$

As it is mentioned in chapter 10 of [8], condition (N) is fulfilled for the smooth and increasing function $\overline{c}(x)$. Moreover as we shall see in Theorem

4.1.2, $W(t,x)$ determines the motion of the wave front for u^{ϵ} for small enough $\epsilon > 0$.

Let us consider $u(t,x)$, the solution to equation (2), for $n = 1$. If we set $u^{\delta}(t,x) = u(t/\delta, x/\delta)$, then u^{δ} is the solution to the following parabolic problem:

$$\begin{aligned} u^{\delta}_{t} &= \frac{\delta}{2}u^{\delta}_{xx} + \frac{\delta}{2}\frac{V_x(x)}{V(x)}u^{\delta}_{x} + \frac{1}{\delta}\overline{c}(x, u^{\delta}(t,x))u^{\delta}, \text{ in } (0,\infty)\times\mathbb{R}^1 \\ u^{\delta}(0,x) &= f(x) \geq 0, \text{ on } \{0\}\times\mathbb{R}^1. \end{aligned} \tag{38}$$

Under the assumptions above, the following theorem, which is a reformulation of Theorem 6.2.1 of [4], states that $W(t,x)$ determines the motion of the wave front for $u^{\delta}(t,x)$ under condition (N):

Theorem 4.1.1. Let $u^{\delta}(t,x)$ be the solution to (38). Then under condition (N) we have:

$$\lim_{\delta\downarrow 0} u^{\delta}(t,x) = \begin{cases} 1, & W(t,x) > 0 \\ 0, & W(t,x) < 0. \end{cases} \tag{39}$$

Let us consider now equation (1) for $n = 1$, $c(\cdot,u) = c(\delta x, u)$, $f(\cdot) = f(\delta x)$ in a slowly changing in x narrow domain $D^{\epsilon,\delta}$, so that $V(\cdot) = V(\delta x)$, $S(\cdot) = S(\delta x)$. Let us define $u^{\epsilon,\delta}(t,x,y) = u^{\epsilon}(t/\delta, x/\delta, y)$. Under the assumptions above, Theorems 3.4 and 4.1.1 imply that $W(t,x)$ will determine the motion of the wave front in this case too, as follows:

Theorem 4.1.2. The following statement holds:

$$\lim_{\delta\downarrow 0}\lim_{\epsilon\downarrow 0} u^{\epsilon,\delta}(t,x,y) = \begin{cases} 1, & W(t,x) > 0 \\ 0, & W(t,x) < 0. \end{cases} \tag{40}$$

So the equation $W(t,x) = 0$ defines the position of the interface (wavefront) between areas where $u^{\epsilon,\delta}$ (for $\epsilon > 0$ and $\delta > 0$ small enough) is close to 0 and to 1. Actually, as we shall see below the wavefront may have jumps. It is known (see chapter 6 in [4]), that because of the dependance of $\overline{c}(x)$ on x, the wave front of u^{δ} may have jumps and new sources may be "igniting" ahead of the front. We will give sufficient conditions that guarantee such jumps for a class of smooth and increasing functions $\bar{c}(x)$. Hence Theorem 4.1.2 implies that one can predict appearances of new sources and jumps of the wave front of $u^{\epsilon,\delta}$ for $\epsilon > 0$ and $\delta > 0$ small enough. Reaction-Diffusion Eqautions (RDE's) with Nonlinear Boundary Conditions In Narrow Domains Let $t^{*} = t^{*}(x, \bar{c}(\cdot))$ be such that $W(t^{*}, x) = 0$. Such a $t^{*}(x, \bar{c}(\cdot))$ is defined in a unique way.

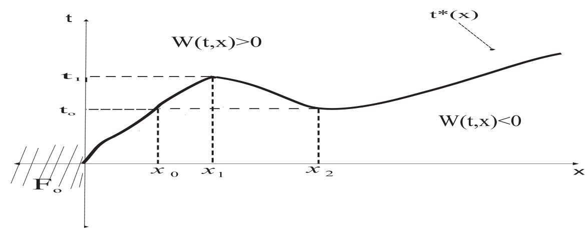

Figure 1: A wavefront that jumps from x_0 to x_2 at time t_0.

We have the following proposition (see chapter 6 in [4] for more details):

Proposition 4.1.3. Let $t^*(x)$ be as in Figure 1 and $F_o = \{x \in \mathbb{R}^1, x < 0\}$. Then the wavefront jumps from x_o to x_2 at time t_o (see Figure 1), i.e.:

(i). If $t \leq t_0$ then $\lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} u^{\epsilon,\delta}(t,x,y) = 1$ for a connected set:
$F_t = \{x \in \mathbb{R}^1 : W(t,x) > 0 \text{ and } x < x_0\}$.

(ii). If $t_0 < t < t_1$ then the set where $\lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} u^{\epsilon,\delta}(t,x,y) = 1$ consists of two connected components:
$F_t = \{x \in \mathbb{R}^1 : W(t,x) > 0 \text{ and } x < x_1\} \cup \{x \in \mathbb{R}^1 : W(t,x) > 0 \text{ and } x > x_1\}$.
The set $\{x \in \mathbb{R}^1 : W(t,x) > 0 \text{ and } x < x_1\}$ is at a positive distance from the set $\{x \in \mathbb{R}^1 : W(t,x) > 0 \text{ and } x > x_1\}$ for $t_0 < t < t_1$.

(iii). If $t \geq t_1$ then $\lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} u^{\epsilon,\delta}(t,x,y) = 1$ for a connected set:
$F_t = \{x \in \mathbb{R}^1 : W(t,x) > 0\}$.

Based now on comparison results (Lemma 4.1.4) we will give sufficient conditions that guarantee jumps of the wavefront. In particular we will prove (Theorem 4.1.5) that if $\overline{c}(x)$ is a rapidly increasing smooth function, then $t^* = t^*(x, \bar{c}(\cdot))$ such that $W(t^*, x) = 0$ is as in Figure 1.

The functional $R_{0,T}(\phi)$ defined in (36) and the function $W(t,x)$ defined in (37) depend also on $\bar{c}$. Hence we will write sometimes $R_{0,T}(\phi, \bar{c}(\cdot))$ and $W(t, x, \bar{c}(\cdot))$ in order to emphasize this dependence.

We have the following comparison result:

Lemma 4.1.4.

(i). Let A be a positive number. Then $t^*(x, Ac(\cdot)) = \frac{1}{\sqrt{A}} t^*(x, c(\cdot))$.

(ii). Let a be a positive number and define $c_a(x) = c(ax)$. Then $t^*(x, c_a(\cdot)) = \frac{1}{a} t^*(ax, c(\cdot))$.

(iii). Let c_1, c_2 be two functions such that $c_1(x) < c_2(x)$ for every $x \in \mathbb{R}^1$. Then $t^*(x, c_1(\cdot)) > t^*(x, c_2(\cdot))$.

Proof. Let us write $t^*_A = t^*(x, Ac(\cdot))$ and let ϕ^A be the extremal so that $W(t^*_A, x, Ac(\cdot)) = R_{0,t^*_A}(\phi^A, Ac(\cdot)) = 0$. Such an extremal satisfies the following Euler-Lagrance equation:

$$\begin{aligned} \ddot{\phi}^A(s) &= -Ac'(\phi^A(s)) \\ \phi^A(0) &= x \\ \phi^A(t^*_A) &= 0. \end{aligned} \tag{41}$$

Let us define now the function $\phi(s) = \phi^A(s/\sqrt{A})$. We claim that the function $\phi(s)$ is the extremal so that $W(\sqrt{A}t^*_A, x, c(\cdot)) = R_{0,\sqrt{A}t^*_A}(\phi, c(\cdot)) = 0$. Indeed it is easy to see that the definition of ϕ and the fact that $R_{0,t^*_A}(\phi^A, Ac(\cdot)) = 0$ imply that $R_{0,\sqrt{A}t^*_A}(\phi, c(\cdot)) = 0$. Moreover ϕ satisfy an Euler-Lagrange equation of the form (41) with $Ac(x)$ and t^*_A replaced by $c(x)$ and $\sqrt{A}t^*_A$ respectively. This proves the claim, which implies part (i) of the lemma.

Part (ii) of the lemma can be proven in a similar way. We define $t^*_a = t^*(x, c_a(\cdot))$ and let ϕ^a to be the extremal so that $W(t^*_a, x, c_a(\cdot)) = R_{0,t^*_a}(\phi^a, c_a(\cdot)) = 0$. Then similarly as it is done in part (i), one should consider the function $\phi(s)$ that is defined by $\phi(s) = a\phi^a(s/a)$.

We prove now part (iii) of the lemma. Let us define $t^*_1 = t^*(x, c_1(\cdot))$ and $t^*_2 = t^*(x, c_2(\cdot))$. Moreover let ϕ^1 be the extremal so that $W(t^*_1, x, c_1(\cdot)) = R_{0,t^*_1}(\phi^1, c_1(\cdot)) = 0$. Since $c_1(x) < c_2(x)$ we have

$$0 = R_{0,t^*_1}(\phi^1, c_1(\cdot)) < R_{0,t^*_1}(\phi^1, c_2(\cdot)). \tag{42}$$

Furthermore, it is easy to see that $W(t, x)$ is an increasing function of t.

Let us assume now that $t^*_1 \leq t^*_2$. This assumption and the fact that $W(t^*_2, x, c_2(\cdot)) = 0$ imply that $W(t^*_1, x, c_2(\cdot)) \leq 0$. By recalling the definition of function W, one easily concludes that:

$$R_{0,t^*_1}(\phi^1, c_2(\cdot)) \leq 0. \tag{43}$$

However inequality (43) contradicts (42). Therefore $t^*(x, c_1(\cdot)) > t^*(x, c_2(\cdot))$.

□

In section 6.2 of [4], it is proven that if $\bar{c}(x)$, instead of the smooth function $\frac{1}{2}\frac{S(x)}{V(x)}c(x,0,0)$, is a piecewise constant function, denoted by $d(x)$, such that

$$d(x) = \begin{cases} d_1, & x < x_2 \\ d_2, & x \geq x_2. \end{cases} \tag{44}$$

with $d_2 > 2d_1 > 0$, then the function $t^* = t^*(x, d(\cdot))$ such that $W(t^*, x, d(\cdot)) = 0$ is not monotone, as in Figure 1. More specifically the curves connecting the point $(0,0)$ with (x_1, t_1) and (x_1, t_1) with (x_2, t_0) are line segments and for $x > x_2$, $t^* = t^*(x, d(\cdot))$ is the solution to

$$\sup_t \{d_2(t^* - t) + d_1 t - \frac{(x - x_2)^2}{2(t^* - t)} - \frac{x_2^2}{2t}\} = 0.$$

Moreover in this case

$$t_0 = x_2 \frac{\sqrt{2(d_2 - d_1)}}{d_2} \tag{45}$$

$$t_1 = \frac{1}{2\sqrt{2d_1}}(x_2 + \sqrt{2d_1} t_0) \tag{46}$$

We will write $t_0 = t_0(d)$ and $t_1 = t_1(d)$ to emphasize the dependence of t_0 and t_1 on the function $d(x)$.

With the help of the result above and Lemma 4.1.4 we will give sufficient conditions that guarantee jumps of the wavefront of $u^\delta(t,x)$ (and by Theorem 4.1.2 of $u^\epsilon(t,x,y)$ for $\epsilon > 0$ and $\delta > 0$ small enough) for a class of smooth and increasing functions $\bar{c}$.

Let us define the set

$$\Delta = \{(d_1, d_2) \in \mathbb{R}^1_+ \times \mathbb{R}^1_+ : d_2 > 2d_1\}. \tag{47}$$

Theorem 4.1.5. Let $d(x)$ be the step function defined in (44) such that $(d_1, d_2) \in \Delta$. Consider real numbers A and a such that

(i). $a, A > 1$.

(ii). $a\sqrt{A} < \frac{1}{2}[1 + \frac{d_2}{2\sqrt{d_1(d_2 - d_1)}}]$.

Then for any smoothly increasing function $\bar{c}(x)$ such that

$$d(x) < \bar{c}(x) < Ad(ax) \tag{48}$$

the wavefront corresponding to $\bar{c}$ has jumps. In particular the excitation reaches the region $\{x > \frac{x_1}{a} + \delta\}$ before it reaches the point $\frac{x_1}{a}$, where δ is a small enough positive number and $\frac{x_1}{a}$ is as in Figures 2 and 3.

Proof. Let us define $\bar{d}(x) = Ad(ax)$. Since $a, A > 1$, the function d(x) is shifted vertically upwards and horizontally to the left. So we get that $d(x) < \bar{d}(x)$ (see Figure 2).

Parts (i) and (ii) of Lemma 4.1.4 imply that $t^*(x, \bar{d}(\cdot)) = \frac{1}{a\sqrt{A}} t^*(ax, d(\cdot))$. This and part (iii) of Lemma 4.1.4 give that if $\bar{c}$ satisfies (48), then $t^*(x, \bar{c}(\cdot))$ will satisfy (see Figure 3):

$$\frac{1}{a\sqrt{A}} t^*(ax, d(\cdot)) < t^*(x, \bar{c}(\cdot)) < t^*(x, d(\cdot)). \tag{49}$$

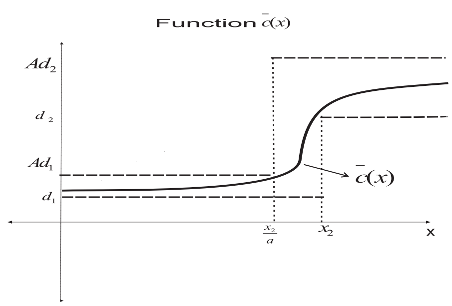

Figure 2: $d(x) < \bar{c}(x) < Ad(ax)$

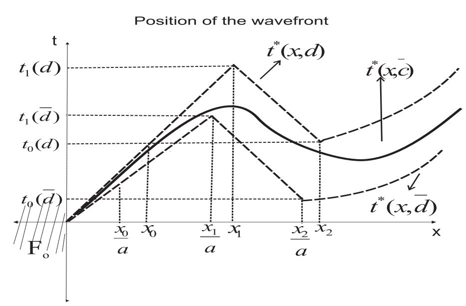

Figure 3: $t^*(x, \bar{d}) < t^*(x, \bar{c}) < t^*(x, d)$

We know that $t^*(x, d(\cdot))$ and $t^*(x, \bar{d}(\cdot))$ are not monotone (recall that d and $\bar{d}$ are piecewise constant functions). We will show that $t^*(x, \bar{c}(\cdot))$ is also not monotone (i.e it is as in Figure 1). Let us assume that

$$t_1(\bar{d}) > t_0(d), \tag{50}$$

where $t_0(d)$ is as in (45) and $t_1(\bar{d})$ is defined similarly to $t_1(d)$ in (46) with d_1, d_2, x_2 replaced by $Ad_1, Ad_2, \frac{x_2}{a}$ respectively. In particular (50) holds if condition (ii) above holds, i.e. if $a\sqrt{A} < \frac{1}{2}[1 + \frac{d_2}{2\sqrt{d_1(d_2-d_1)}}]$. Moreover, it is easy to see that $d_2 > 2\sqrt{d_1(d_2 - d_1)}$ is true for any $d_2 > d_1 > 0$. This implies that $\frac{1}{2}[1 + \frac{d_2}{2\sqrt{d_1(d_2-d_1)}}] > 1$, which has to be true since $a, A > 1$.

Inequality (50) can be equivalently written as $t^*(\frac{x_1}{a}, \bar{d}(\cdot)) > t(x_2, d(\cdot))$. By this and (49) we immediately get that

$$t^*(x_2, \bar{c}(\cdot)) < t^*(\frac{x_1}{a}, \bar{c}(\cdot)) \tag{51}$$

which, since $\frac{x_1}{a} < x_1 < x_2$, implies that $t^*(x, \bar{c}(\cdot))$ is as in Figure 1 and so new sources are igniting ahead of the wavefront.

In Figures 2 and 3 we see an illustration of the construction.

□

Example. An example of a function $\bar{c}(x)$ that satisfies the requirements of Theorem 4.1.5 is

$$\bar{c}(x) = \frac{Ad_2\mu + d_1 e^{-\lambda(x-k)}}{\mu + e^{-\lambda(x-k)}}, \tag{52}$$

where $(d_1, d_2) \in \Delta$, a, A satisfy assumptions (i) and (ii) of Theorem 4.1.5, $k \in (\frac{x_2}{a}, x_2)$ and the constants μ and λ are chosen so that $\bar{c}(\frac{x_2}{a}) < Ad_1$ and $\bar{c}(x_2) > d_2$.

In particular now if $\overline{c}(x) = \frac{1}{2}\frac{S(x)}{V(x)}$, i.e. $c(x,0,0) = 1$, is an increasing smooth function that satisfies the requirements of Theorem 4.1.5, then the jump of the wavefront of $u^{\epsilon,\delta}(t,x,y)$, for $\epsilon > 0$ and $\delta > 0$ small enough, occurs when $\frac{S(x)}{V(x)}$ increases rapidly. This implies, at least when the tube D^1 retains its shape as x increases, that the jumps of the wave front occur at places where the tube D^1 becomes thinner, i.e. when $V(x)$ decreases significantly.

Remark 4.1.6. Similar results hold for layers as well, i.e. for $x \in \mathbb{R}^n$ with $n > 1$.

Using the results in [7] one can consider the limiting behavior as $\delta, \epsilon \downarrow 0$ of $u^{\epsilon,\delta}(t,x,y)$ when condition (N) is not fulfilled. We will briefly discuss the result for the general case $x \in \mathbb{R}^n$.

Instead now of function $W(t,x)$ defined by (37), we consider the function

$$\begin{aligned} W^*(t,x) = \sup\{\min_{0\le s\le t} R_{0,s}(\phi) \quad &: \quad \phi \in \mathcal{C}_{0,t}(\mathbb{R}^n) \text{ is absolutely continuous,} \\ &\quad\ \phi_0 = x, \quad \phi_t \in F_o\}. \end{aligned} \tag{53}$$

One can prove that $W^*(t,x)$ is Lipschitz continuous and that $W^*(t,x) \le \min\{0, W(t,x)\}$.

Then Theorem 2.1 in [7] and Theorem 3.4 imply that $W^*(t,x)$ determines the motion of the wave front as follows:

Theorem 4.1.7. The following statements hold:

(i). For any compact subset Θ_1 of the interior of $\{(t,x) : t > 0, W^*(t,x) = 0\}$,

$$\lim_{\delta\downarrow 0}\lim_{\epsilon\downarrow 0} u^{\epsilon,\delta}(t,x,y) = 1 \text{ uniformly in } (t,x) \in \Theta_1.$$

(ii). For any compact subset Θ_2 of $\{(t,x): W^*(t,x)<0\}$,

$$\lim_{\delta\downarrow 0}\lim_{\epsilon\downarrow 0} u^{\epsilon,\delta}(t,x,y)=0 \text{ uniformly in } (t,x)\in\Theta_2.$$

We conclude subsection 4.1 with the case that the nonlinear term $\overline{c}(x,u)$ of (2) is of bistable type, i.e. $\overline{c}(x,u)>0$ for $u\in(\mu,1)$, $\overline{c}(x,u)<0$ for $u\in(0,\mu)\cup(1,\infty)$, where $0<\mu<1$. This problem was considered in [10] and it was also presented in section 6.4 of [4].

Here we restrict the analysis to a concrete example that allows to give an exact formula for the asymptotic speed of the wavefront of $u^{\epsilon,\delta}$ for $\epsilon>0$ and $\delta>0$ small enough. As we will see the asymptotic speed of the wavefront is proportional to the square root of the surface area to volume ratio $\sqrt{\frac{S(x)}{V(x)}}$.

To be specific let $x\in\mathbb{R}^n$, $c(x,0,u)=(u-\mu)(1-u)$, $0<\mu<\frac{1}{2}$ and assume that the function $u^\delta(t,x)$ (compare with (38)) is the solution to

$$\begin{aligned} u^\delta_t &= \frac{\delta}{2V(x)}\mathrm{div}(V(x)\nabla_x u^\delta)+\frac{1}{\delta}\frac{1}{2}\frac{S(x)}{V(x)}(u^\delta-\mu)(1-u^\delta)u^\delta, \quad \text{in } (0,\infty)\times\mathbb{R}^n \\ u^\delta(0,x) &= f(x), \quad \text{on } \{0\}\times\mathbb{R}^n. \end{aligned} \tag{54}$$

Consider a point $x\in\mathbb{R}^n$ to be excited at time t, if $u^\delta(t,x)$ (the solution to (54)) is close to 1 and non-excited if $u^\delta(t,x)$ is close to 0. Then the Corollary of Theorem 4.1 of [10] gives us that for small $\delta>0$ the region $\{x\in\mathbb{R}^n: f(x)>\mu\}$ becomes excited and the region $\{x\in\mathbb{R}^n: f(x)<\mu\}$ becomes non-excited after a short starting phase. Now let $u^{\epsilon,\delta}(t,x,y)=u^\epsilon(t/\delta,x/\delta,y)$, where $u^\epsilon(t,x,y)$ is the solution to (1). Theorem 3.4 implies that the same conclusions hold for $u^{\epsilon,\delta}(t,x,y)$ for $\epsilon>0$ and $\delta>0$ small enough.

To compute the asymptotic propagation speed of excitation at $x\in\mathbb{R}^n$, let us consider the equation for the wave profile:

$$\begin{aligned} &\frac{1}{2}v''_{\xi\xi}(\xi)+a(x)v'_\xi(\xi)+\frac{1}{2}\frac{S(x)}{V(x)}(v(\xi)-\mu)(1-v(\xi))v(\xi)=0, \quad \xi\in\mathbb{R} \\ &\lim_{\xi\to-\infty}v(\xi)=1, \quad \lim_{\xi\to\infty}v(\xi)=0. \end{aligned} \tag{55}$$

As it can be verified by direct substitution, equation (55) is solvable if $a(x)$ is given by the formula

$$a(x)=\sqrt{\frac{1}{2}\frac{S(x)}{V(x)}}\left(\frac{1}{2}-\mu\right). \tag{56}$$

Moreover, in our case, (56) is also the asymptotic propagation speed of excitation at $x\in\mathbb{R}^n$ and it is independent of direction.

Lastly, it is known that as the size of D_x increases (without changing shape), the surface area to volume ratio $\frac{S(x)}{V(x)}$ decreases. In the case $x \in \mathbb{R}$, this fact, equation (56) and Theorem 3.4 imply that the wavefront of $u^{\epsilon,\delta}$ (for $\epsilon > 0$ and $\delta > 0$ small enough) slows down when the tube becomes thicker. A similar result holds for layers.

4.2 K-P-P Fronts in Random Media

In this subsection we consider wave front propagation for the solution of (1) for small $\epsilon > 0$, when $x \in \mathbb{R}$, the boundary ∂D^1 of D^1 is determined by stationary and ergodic random processes on $\mathbb{R}$ and the nonlinear boundary term in (1) (for $y = 0$, i.e. $c(x, 0, u)$) is of K-P-P type. As we did in subsections 4.1, we will first consider (Theorem 4.2.6) wavefront propagation for the solution of (2) and then with the aid of Theorem 3.4 we will consider (Theorem 4.2.7) wavefront propagation for the solution of (1) for small enough $\epsilon > 0$. As we will see the cross sections D_x of D affect the speed of the wavefront through the surface to volume ratio $\frac{S(x)}{V(x)}$.

In sections $7.4 - 7.6$ of [4] wave front propagation for equations like (2) is considered in the case where there is no drift term and the randomness comes only from the nonlinear part of the equation. Moreover in [13] the authors considered the case of reaction-diffusion equations of type (2) with a random drift and homogeneous in x nonlinear term. In the case considered here, both the drift and and the nonlinear term are random. In [4], pp. 524-525, the author remarks that one could use the procedure developed in sections $7.4 - 7.6$ of [4] to study wavefronts in one-dimensional uniformly bounded random drift with random nonlinear term. We will see that one can prove Theorem 4.2.6, which is analogous to Theorem 7.6.1 in [4], by following the proof of Theorem 7.6.1 in [4]. We make use of the results in [13] and of the fact that the operator of the equation (2) is self adjoint with respect to an appropriate inner product (it has the form $\frac{1}{2V(x)}\frac{d}{dx}(V(x)\frac{d}{dx})$). Actually the latter simplifies the analysis significantly. Instead of repeating the proof of [4] here, we will only outline the differences.

Let us first list our assumptions. Consider a probability space $(\hat{\Omega}, \hat{\mathfrak{F}}, \hat{P})$. We assume that the random field $V(x, \hat{\omega})$ (namely the volume) is three times continuously differentiable, i.e. $V \in \mathcal{C}^3(\mathbb{R})$, with $\hat{P}$ probability one. Suppose that $\Theta(x) = (\frac{d}{dx}(\log V(x)), \frac{S(x)}{V(x)})$ is a random vector function on $(\hat{\Omega}, \hat{\mathfrak{F}}, \hat{P})$ and that it is measurable, stationary in x and translation in x generates an ergodic transformation of the space $\hat{\Omega}$. Moreover the function $\frac{d}{dx}(\log V(x))$ is assumed bounded, with zero mean (i.e. $\hat{E}[\frac{d}{dx}(\log V(x))] = 0$). We additionally assume (for the purposes of Lemma 4.2.1 and 4.2.3) that there is a set of nonzero $\hat{P}$ probability on which

$$\lim_{z\to\infty}\int_0^z [V(x,\hat{\omega})]^{-1}dx = +\infty. \tag{57}$$

If condition (57) holds on a set of nonzero measure then, by the ergodicity assumption, it must hold with $\hat{P}$ probability one.

As far as the non-linear term $\overline{c}(x,u,\hat{\omega})u = \frac{S(x,\hat{\omega})}{V(x,\hat{\omega})}c(x,0,u)u$ is concerned, in addition to the stationarity and ergodicity assumptions, we also make the following assumptions. For all $x \in \mathbb{R}$, c is of K.P.P type, i.e. $c(x,0,u)$ is positive for $u < 1$, negative for $u > 1$, continuous in u for $u \geq 0$ and $c(x) = c(x,0,0) = \sup_{0<u} c(x,0,u)$. Moreover with $\hat{P}$ probability one, the function $\overline{c}(x,u,\hat{\omega})u$ satisfies a Lipschitz condition of the form

$$|\overline{c}(x,u_1,\hat{\omega})u_1 - \overline{c}(x,u_2,\hat{\omega})u_2| \leq \frac{S(x,\hat{\omega})}{V(x,\hat{\omega})}\zeta(x)|u_1-u_2|, \text{ for } x,u_1,u_2 \in \mathbb{R},$$

such that for all $t \geq 0$ and $x \in \mathbb{R}$,

$$E_x \exp\{\int_0^t \frac{S(X_s)}{V(X_s)}\zeta(X_s)\} < \infty, \quad \hat{P}\text{-a.s.},$$

where (X_t, P_x) is a diffusion process with random generator $L = \frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}\frac{d}{dx}(\log V(x,\hat{\omega}))\frac{d}{dx}$.

The initial function $f(x)$ is assumed to be nonnegative, bounded from above and non-random.

Let now $\mu(z)$ be the function defined by the equality

$$\mu(z) = \hat{E}[\ln E_1 \chi_{\tau_0<\infty} \exp\{\int_0^{\tau_0}[\overline{c}(X_s)+z]ds\}], \quad z \in \mathbb{R}, \tag{58}$$

where $\bar{c}(x) = \frac{1}{2}\frac{S(x)}{V(x)}c(x,0,0)$ and τ_0 is the first hitting time of the process X_t to the point 0. For τ_0 one has the following lemma:

Lemma 4.2.1. Condition (57) and $\hat{E}[\frac{d}{dx}(\log V(x,\hat{\omega}))] = 0$ imply that $P_1(\tau_0 < \infty) = 1$.

Proof. It follows directly from the proof of Lemma 4.4 of [13] if one notes that the drift term is $\frac{1}{2}\frac{d}{dx}(\log V(x))$.

□

Lemma 4.2.2. Under the assumptions imposed above, function $\mu(z)$ has the following properties:

(i). For all $z \in \mathbb{R}$, $\mu(z) = \lim_{t\to\infty} \frac{1}{t} \ln E_t \chi_{\tau_0<\infty} \exp\{\int_0^{\tau_0} [\bar{c}(X_s) + z] ds\}$.

(ii). Function $\mu(z)$ is convex, lower semicontinuous and monotonically non-decreasing in z. Moreover $\mu(z)$ is continuously differentiable and the derivative $\mu'(z)$ is positive and monotonically increasing for $z < \bar{g}_\mu$, where $\bar{g}_\mu$ is a non-positive number (which actually is the discontinuity point of $\mu(z)$, as property (iii) below shows).

(iii). $\mu(z) \leq 0$ for $z \leq \bar{g}_\mu$ and $\mu(z) = \infty$ for $z > \bar{g}_\mu$ where $\bar{g}_\mu \leq 0$.

Proof. Property (i) can be proven as Proposition 2.1 of [13]. Property (ii) follows similarly as Theorem 7.5.1(ii) of [4]. Property (iii) follows analogously to Theorem 7.5.1(iii) of [4]. Here one uses the fact that the operator of (2) has the form $\frac{1}{2V(x)}\frac{d}{dx}(V(x)\frac{d}{dx})$), i.e. it is self adjoint.

□

We also observe that $\mu(z) \geq \mu_o(z)$ where $\mu_o(z) = \hat{E}[\ln E_1(\chi_{\tau_0<\infty} e^{z\tau_0})]$. As it has been proven in Lemma 2.2 of [13], function $\mu_o(z)$ has properties (i)-(iii) of Lemma 4.2.2 as well (for $\bar{c}(x) = 0$). In addition the following lemma holds, which is a restatement of Proposition 4.1 of [13].

Lemma 4.2.3. Condition (57) and $\hat{E}[\frac{d}{dx}(\log V(x, \hat{\omega}))] = 0$ imply that the discontinuity point of $\mu_o(z)$ is $\bar{g}_{\mu_o} = 0$.

We will assume that $-\infty < \bar{g}_\mu < 0$ (by Lemma 4.2.2(iii) or Lemma 4.2.3 we already know that $\bar{g}_\mu \leq 0$) and we define $I(y) = \sup_{z\leq \bar{g}_\mu}[yz - \mu(z)]$ for $y \in \mathbb{R}$.

Lemma 4.2.2 and the fact that $\mu(z) \geq \mu_o(z)$ imply that the arguments in the beginning of section 7.6 of [4] carry out here as well. Therefore we conclude that there is a unique $\nu^* > 0$ such that $I(\frac{1}{\nu^*}) = 0$ and $\nu^* = \inf_{z\leq \bar{g}_\mu} \frac{z}{\mu(z)}$.

Remark 4.2.4. We would like to emphasize that the existence and uniqueness of a positive ν^* follows mainly from properties (i)-(iii) of $\mu(z)$ (Lemma 4.2.2). In particular property (iii) holds because the operator of (2) is self adjoint.

Similarly as Theorem 7.6.1 in [4] was proven, one can prove Theorem 4.2.6 below.

Note that by following the proof of Theorem 7.6.1 in [4], one needs to estimate certain probabilities for τ_0 and X_t. For this purpose we have the following lemma:

Lemma 4.2.5. Let δ be a positive number and $U_\delta(0) = \{x : |x| \leqslant \delta\}$. Then (i). $\inf_{x\in U_\delta(0)} P_x\{\tau_0 \leqslant 1\} > 0$, $\hat{P}$-a.s.

(ii). $\inf_{x\in U_\delta(0), s\in(0,1]} P_x\{X_s \in U_\delta(0)\} > 0$, $\hat{P}$-a.s.

(iii). For $a > 0$ and $\eta > \delta > 0$ we have

$$\inf_{x\in U_\delta(-a)} P_x\{\tau_{-\eta-a} > 1, X_1 \in U_\delta(0)\} > 0,\ \hat{P}\text{-a.s.}$$

Proof. The proof of all statements follows from the corresponding statements for W_t^1 in place of X_t (see for example section 7.5 of [4]) and by the Girsanov's theorem on the absolute continuous change of measures in the space of trajectories.

□

Therefore we have the following Theorem:

Theorem 4.2.6. Let $x \in \mathbb{R}$ and $u(t,x)$ satisfy equation (2). Under our assumptions we have:

(i). For all $\nu > \nu^*$,

$$\lim_{t\to\infty} \sup_{x\geq\nu t} u(t,x) = 0, \quad \hat{P} - a.s.$$

(ii). Let us define $\bar{c}_h(x) = \frac{1}{2}\frac{S(x)}{V(x)} \inf_{0<u<h} c(x,0,u)$ and assume that there is a constant $\kappa > 0$ such that for any $0 < h < 1$ and $x \in \mathbb{R}$,

$$\kappa < \bar{c}_h(x), \quad \hat{P} - a.s.$$

Then for all $\nu \in (0, \nu^*)$,

$$\lim_{t\to\infty} \sup_{0\leq x\leq\nu t} u(t,x) = 1, \quad \hat{P} - a.s.$$

Finally Theorem 3.4 and Theorem 4.2.6 imply:

Theorem 4.2.7. Let $(x,y) \in \mathbb{R}\times\mathbb{R}^m$ and $u^\epsilon(t,x,y)$ satisfy equation (1). Under our assumptions we have:

(i). For all $\nu > \nu^*$,

$$\lim_{t\to\infty} \sup_{x\geq\nu t} \lim_{\epsilon\to 0} u^\epsilon(t,x,y) = 0, \quad \hat{P} - a.s.$$

(ii). Let us define $\bar{c}_h(x) = \frac{1}{2}\frac{S(x)}{V(x)} \inf_{0<u<h} c(x,0,u)$ and assume that there is a constant $\kappa > 0$ such that for any $0 < h < 1$ and $x \in \mathbb{R}$,

$$\kappa < \bar{c}_h(x), \quad \hat{P} - a.s.$$

Then for all $\nu \in (0, \nu^*)$,

$$\lim_{t\to\infty} \sup_{0\leq x\leq \nu t} \lim_{\epsilon\to 0} u^\epsilon(t,x,y) = 1, \quad \hat{P} - a.s.$$

Remark 4.2.8. Theorem 4.2.6 was proven in ([4]) with the assumption in part (ii) replaced by the assumption that for any $0 < h < 1$ and $\nu \in \mathbb{R}$,

$$\limsup_{t\to\infty} \frac{1}{t} \ln E_{\nu t} \exp\{-\int_0^t \bar{c}_h(X_s)ds\} < 0, \quad \hat{P} - a.s, \tag{59}$$

which is however difficult to verify. Obviously the assumption made in part (ii) of Theorems 4.2.6 and 4.2.7 implies (59).

5 Acknowledgements

M. Freidlin was partially supported by the NSF. The authors would like to thank Prof. K. Trivisa and Prof. M. Grillakis for helpfull discussions.

References

[1] M. Bochniak, Linear elliptic boundary value problems in varying domains, Math. Nachr. 250 (2003), pp. 17-24.

[2] S.N. Ethier, T.G. Kurtz, Markov processes: Characterization and Convergence, Wiley, New York, 1986.

[3] L.C. Evans, P.E. Souganidis, A PDE approach to geometric optics for certain semilinear parabolic equations, Indiana U. Math, Vol. 52 (1989), pp. 43-80.

[4] M. Freidlin, Functional Integration and Partial Diffferential Equations, Princeton University Press, 1985.

[5] M. Freidlin, Wave Front Propagation for KPP-Type Equations, Survey in Applied Mathematics, 2 (1995), pp. 1-62.

[6] M. Freidlin, Markov Processes and Differential Equations: Asymptotic Problems, Birkhäuser-Verlag, Basel, Boston, Berlin, 1996.

[7] M. Freidlin, Coupled Reaction Diffusion Equations, Annals of Probability, Vol. 19, No. 1 (1991), pp. 29-57.

[8] M.I. Freidlin, A.D. Wentzell, Random Perturbations of Dynamical Systems, 2nd edn, Springer-Verlag, New York, 1998.

[9] A. Friedman, Partial Differential Equations of Parabolic Type, Prentice Hall, Englewood Cliffs, NJ, 1964.

[10] J. Gärtner, Bistable Reaction-Diffusion Equations and Excitable Media, Mathematiche Nachrichten, Vol. 112 (1983), pp. 125-152.

[11] I.Karatzas, S.E.Shreve, Brownian Motion and Stochastic Calculus, 2nd edn, Springer-Verlag, New York, 1994.

[12] A. Kolmogorov, I. Petrovskii, N. Piskunov, Étude de l′èquation de la diffusion avec croissence de la matière et son application a un problème biologique, Moscov University Bull. Math., Vol. 1 (1937), pp. 1-25.

[13] J. Nolen, J. Xin, KPP Fronts in a One-Dimensional Random Drift, Discrete and Continuous Dynamical Systems-B, to appear (2007),

[14] W. Pogorzelski, Propriétés des intégrales de l'équation parabolique normale, Annales Polonici Mathematici, Vol. 4 (1957), pp. 61-92.

[15] W. Pogorzelski, Propriétés des dérivées tangentielles d'une intégrale de l'équation parabolique, Ricerche di Matematica, Vol. 6 (1957), pp. 162-194.

[16] W. Pogorzelski, Propriétés aux limites aux des dérivées tangentielles pour l'équation parabolique, Annales scientifiques de l'École Normale Supérieure Sér. 3, 75 no. 1 (1958), pp. 19-35.