Nonlinear convective stability of travelling fronts near Turing and Hopf instabilities

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Joint work with
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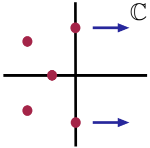
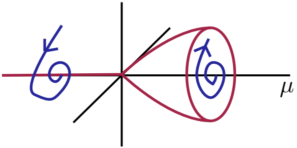
September 28, 2009
Boston University Dynamics Seminar

Motivation: Ways to have a Hopf bifurcation

ODEs: Hopf bifrucation via eigenvalues



Supercritical Hopf
as μ increases

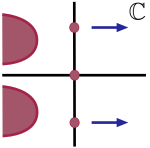


Dynamics on
$3D$ center manifold

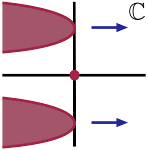
- Before bifurcation: $u(x) = 0$ stable
- After bifurcation: $u(x) = 0$ unstable
- After bifurcation: nearby stable periodic orbit exists
- After bifurcation: solutions approach periodic orbit, amplitude saturates

Motivation: Ways to have a Hopf bifurcation

PDEs: Hopf bifurcation via eigenvalues or essential spectrum



Similar to above

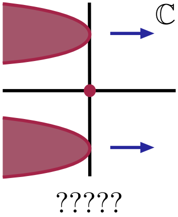


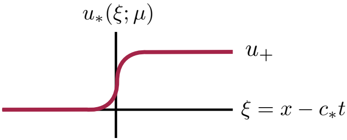
?????

- Before bifurcation: equilibrium is stable
- After bifurcation: equilibrium is linearly unstable
- After bifurcation: do nearby stable solutions exist?
- After bifurcation: what state do solutions approach?

"Essential" Hopf bifurcations

What causes an "essential" Hopf bifurcation?





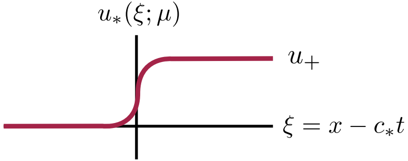
Roughly speaking:

- Eigenvalues: localized perturbations and interior of wave
- Essential spectrum: non-localized perturbations and end states of wave

An "essential" Hopf bifurcation is caused by a destabilization of the end states.

Essential instabilities of fronts: results of Sandstede and Scheel '01

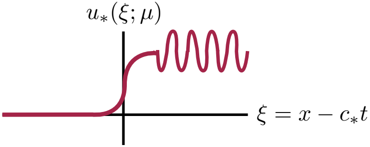
What patterns can form through essential Hopf instabilities of fronts?

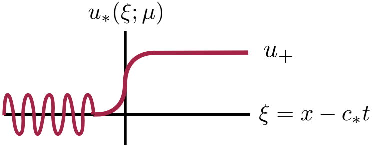


Two cases:

- Rest state ahead of front destabilizes: u_+
- Rest state behind front destabilizes: 0

Expect fronts connecting remaining stable state to emergent patterns:





Essential instabilities of fronts: results of Sandstede and Scheel '01

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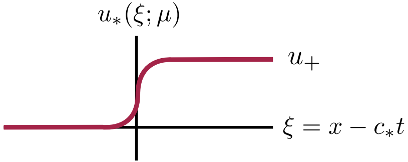

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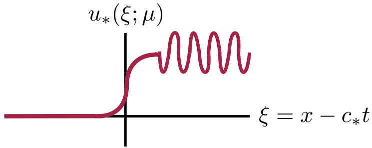
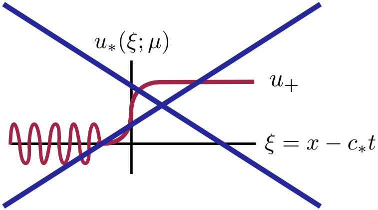
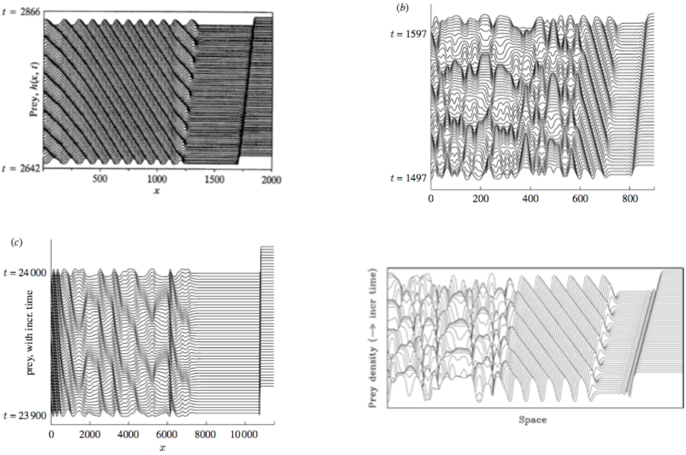




Essential instabilities of fronts: ecological examples

Patterns in wake of front in predator-prey models (matches field studies):

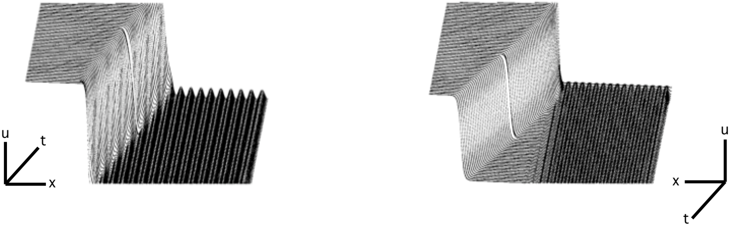


[From papers by Jonathan A. Sherratt and colleagues]

In all cases the front outruns the pattern.

Essential instabilities of fronts: results of Sandstede and Scheel '01

- Destabilization ahead: patterns exist and are stable!
- Destabilization behind: patterns do not exist; front outruns it!



[From Sandstede and Scheel, Dynamical Systems Vol. 16, 2001]

Sandstede and Scheel explain this using exponential dichotomies and Fredholm theory: roughly speaking, dimension counting of stable and unstable manifolds.

Question: when no emergent pattern exists, what is the "stable" behavior?

- Front becomes linearly unstable
- Front is still be observed: must be nonlinearly stable

Set up and assumptions

Reaction-diffusion system

$$u_t = D\partial_x^2 u + f(u;\mu), \quad u \in \mathbb{R}^n, \quad x \in \mathbb{R}, \quad \mu \approx 0 \qquad \text{(RD)}$$

Hypothesis 1: *Existence of front solution* $\forall \mu \approx 0$

$$u(x,t) = u_*(x - c(\mu)t;\mu), \qquad c(\mu) > 0$$
$$\lim_{\xi \to -\infty} u_*(\xi;\mu) = 0, \qquad \lim_{\xi \to +\infty} u_*(\xi;\mu) = u_+$$

Linearized operator at critical front:

$$\mathcal{L}_* = D\partial_\xi + c_*\partial_\xi + f_u(u_*^0(\xi);0), \qquad \xi = x - c_* t, \quad c_* = c(0)$$

Asymptotic operators:

$$\mathcal{L}_-(\mu) = D\partial_x^2 + f_u(0;\mu), \qquad \mathcal{L}_+(0) = D\partial_x^2 + f_u(u_+;0),$$

Exponential weight:

$$\rho_a(\xi) = \begin{cases} 1 & \text{if } \xi \geq 1 \\ e^{a\xi} & \text{if } \xi \leq -1 \end{cases}$$

Set up and assumptions

Hypothesis 2: *Spectral assumptions*

- $0 < a \leq a_0$: *the spectrum of* $\mathcal{L}_*^a := \rho_a \mathcal{L}_* \rho_a^{-1}$ *is in open left half plane except isolated eigenvalue at* 0.
- *For* $\mu \approx 0$, *spectrum of* $\mathcal{L}_-(\mu)$ *is in open left half plane except for*

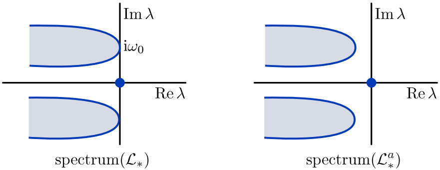
$$\lambda(k,\mu) = \lambda_0(\mu) - \lambda_2(\mu)(k-k_0)^2 + \mathcal{O}(|k-k_0|^3), \qquad |k-k_0| \ll 1$$

and its complex conjugate, where $\mathrm{Re}\,\lambda_2(0) > 0$, $\mathrm{Re}\,\lambda_0'(0) > 0$ *and either*

Turing: $k_0 > 0$ *and* $\lambda_0(0) = 0$, *or*

Hopf: $k_0 = 0$, $\lambda_0(0) = \mathrm{i}\omega_0$ *for some* $\omega_0 > 0$.

- *The spectrum of* $\mathcal{L}_+(0)$ *lies in the open left half plane.*



Note: Picture is in moving frame ξ, so $\omega_0 = k_0 c_* > 0$ at Turing bifurcations.

Set up and assumptions

Need bifurcation to be supercritical:

$$u(x,t) = \epsilon e^{\mathrm{i}(k_0 x + \omega_0 t)} A(\epsilon x, \epsilon^2 t) e(k_0) + \mathrm{c.c.}, \qquad \mu = \rho \epsilon^2$$

Amplitude $A(X,T)$ satisfies Ginzburg-Landau equation

$$A_t = \lambda_2(0)\partial_X^2 A + \rho\lambda_0'(0)A - b|A|^2 A$$

Hypothesis 3: *Supercritical bifurcation*

$$\mathrm{Re}\, b > 0$$

This ensures the growth of the emergent pattern saturates.

Function space: uniformly local functions

$$\rho_{\mathrm{ul}}(x) = e^{-|x|}, \qquad \|u\|_{\rho_{\mathrm{ul}}}^2 = \int_{\mathbb{R}} \rho_{\mathrm{ul}}(x)|u(x)|^2 \,\mathrm{d}x \qquad \|u\|_{L^2_{\mathrm{ul}}} = \sup_{y\in\mathbb{R}} \|u\|_{T_y \rho_{\mathrm{ul}}}.$$

Like normal Sobolev spaces but allow for nonlocalized functions.

Statement of result

Theorem [B., Ghazaryan, Sandstede JDE 09] *Assume* **(H1)-(H3)**, *then there exist positive constants* K, Λ_*, a_*, μ_*, *and* δ_* *such that: for any*

$$|\mu| \leq \mu_*, \qquad \|v(\cdot,0)\|_{H^1_{\mathrm{ul}}} < \delta_*,$$

the solution of (RD) with $u(x,0) = u_*(x;\mu) + v(x,0)$ *exists for all* $t \geq 0$ *and*

$$u(x,t) = u_*(x - c(\mu)t - p(t);\mu) + v(x - c(\mu)t, t)$$

for an appropriate real-valued function p*; furthermore,* $\exists\ p_* \in \mathbb{R}$ *such that*

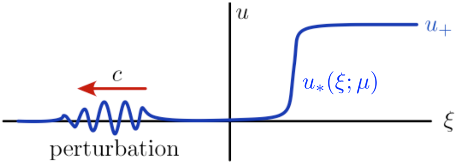
$$\begin{aligned} \|v(\cdot,t)\|_{H^1_{\mathrm{ul}}} + |p(t)| &\leq K\left(\|v(\cdot,0)\|_{H^1_{\mathrm{ul}}} + \sqrt{|\mu|}\right) \\ \|\rho_{a_*}(\cdot)v(\cdot,t)\|_{H^1_{\mathrm{ul}}} + |p(t) - p_*| &\leq K\mathrm{e}^{-\Lambda_* t} \end{aligned}$$

for all $t \geq 0$.

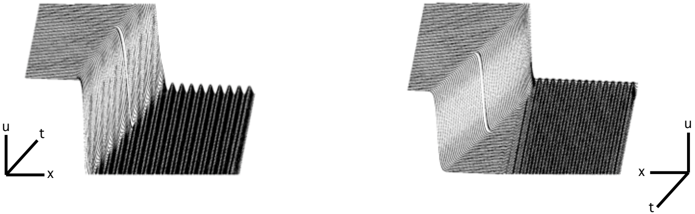
In other words, the perturbation $v(\xi,t)$ *decays to zero exponentially in time in the weighted norm* $\|\rho_{a_*}\cdot\|_{H^1_{\mathrm{ul}}}$ *in the comoving frame* $\xi = x - c(\mu)t$.

Intuition

Front outruns emergent pattern



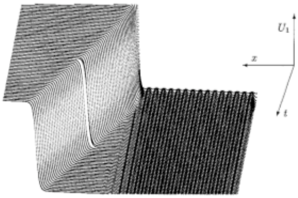
and because bifurcation is supercritical, growth of pattern saturates.

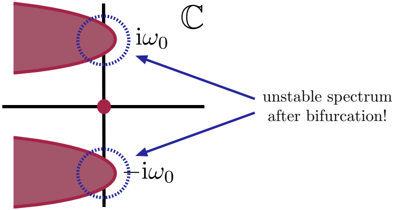


[From Sandstede and Scheel, Dynamical Systems Vol. 16, 2001]

Difficulties in proof

Just after bifurcation:





Mathematical issues:

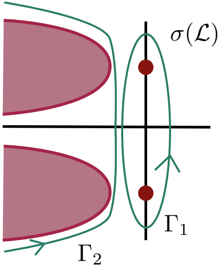
- Need to control the growth this causes
- No spectral gap: how to isolate this growth?

Resolution: mode filters

- Developed by G. Schneider, 1994 papers
- Generalization of a spectral projection

Mode filters

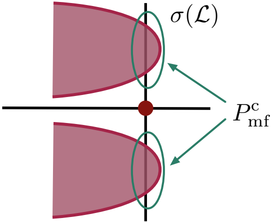
Standard spectral projection:



$$e^{\mathcal{L}t} = \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_1} e^{\lambda t}(\lambda - \mathcal{L})^{-1}\mathrm{d}\lambda + \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_2} e^{\lambda t}(\lambda - \mathcal{L})^{-1}\mathrm{d}\lambda$$

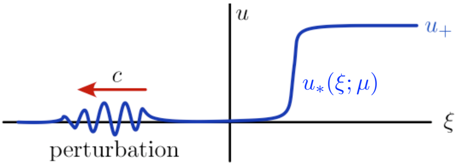
$$= e^{\mathcal{L}t}P^{\mathrm{c}} + e^{\mathcal{L}t}P^{\mathrm{s}}$$

Mode filters effectively allow for:



$$e^{\mathcal{L}t} = e^{\mathcal{L}t}P^{\mathrm{c}}_{\mathrm{mf}} + e^{\mathcal{L}t}\underbrace{(1 - P^{\mathrm{c}}_{\mathrm{mf}})}_{P^{\mathrm{s}}_{\mathrm{mf}}}$$

Proof: two steps



$$u(x,t) = u_*(x - c(\mu)t - p(t);\mu) + v(x - c(\mu)t, t)$$

Step 1: A prior estimates in weighted space:

$$\begin{array}{lll} \text{If} & ||v(t)||_{H^1_{\mathrm{ul}}} < \delta & \forall t \geq 0 \\ \text{Then} & ||\rho_a v(t)||_{H^1_{\mathrm{ul}}} \leq K e^{-\Lambda_* t} & \forall t \geq 0 \end{array}$$

Step 2: Use mode filters to prove that, indeed

$$||v(t)||_{H^1_{\mathrm{ul}}} < \delta \quad \forall t \geq 0$$

Note: strategy similar to [Ghazaryan & Sandstede '07], but they had a specific system and their step 2 was proved using the maximum principle.

Step 1: A priori estimates in weighted space

Expect: $u(\xi, t) \to u_*(\xi - p)$ as $t \to \infty$.

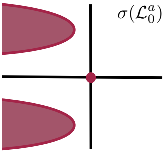
$$u(x,t) = \underbrace{u_*(x - c(\mu)t - p(t);\mu)}_{u_*(\xi - p(t))} + \underbrace{v(x - c(\mu)t, t)}_{v(\xi, t)}$$

Unshifted linearized operator:

$$\mathcal{L}_0 = D\partial_\xi^2 + c(\mu)\partial_\xi + f_u(u_*(\xi))$$

Work in weighted space: $w(\xi, t) := \rho_a(\xi) v(\xi, t)$

$$w_t = \mathcal{L}_0^a w + F(p, \xi) w + N(v, w)$$



Because in weighted space there is a spectral gap:

$$\begin{aligned} w_t &= P^{\mathrm{s}}\left[\mathcal{L}_0^a w + F(p,\xi)w + N(v,w)\right] \\ p_t &= C P^{\mathrm{c}}\left[\tilde{F}(p,\xi)w + N(v,w)\right] \end{aligned}$$

Step 1: A priori estimates in weighted space

Given η sufficiently small, define $T_{\max}(\eta) > 0$ to be the maximum time so that

$$\sup_{t\in[0,T]} \Big(|p(t)| + \|v(\cdot,t)\|_{H^1_{\mathrm{ul}}} \Big) \leq \eta$$

Lemma *There exists a* Λ *so that if* w *is a solution in the weighted space, then*

$$\|w(\cdot,t)\|_{H^1_{\mathrm{ul}}} \leq K\mathrm{e}^{-\Lambda t}\|w(\cdot,0)\|_{H^1_{\mathrm{ul}}}, \qquad |p(t)| \leq K\|w(\cdot,0)\|_{H^1_{\mathrm{ul}}}$$

for all $0 \leq t \leq T_{\max}(\eta)$*, for some positive constant* K *that is independent of* μ *and* η*. If* $T_{\max}(\eta) = \infty$*, then there is a* $p_* \in \mathbb{R}$ *with*

$$|p(t) - p_*| \leq K\mathrm{e}^{-\Lambda t}\|w(\cdot,0)\|_{H^1_{\mathrm{ul}}}$$

for all $t \geq 0$*.*

Therefore, in step 2 we need to show $T_{\max}(\eta) = \infty$.

Step 2: estimates in unweighted space via mode filters

Proposition *There exist positive constants* K, δ_* *and* μ_* *such that, if* $\|v(0)\|_{H^1_{\mathrm{ul}}} < \delta_*$, *then for each* μ *with* $|\mu| \le \mu_*$ *the perturbation satisfies*

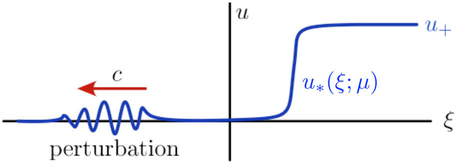
$$\|v(\cdot,t)\|_{H^1_{\mathrm{ul}}} + |p(t)| \le K\left(\|v(\cdot,0)\|_{H^1_{\mathrm{ul}}} + \sqrt{|\mu|}\right)$$

for all $t \ge 0$. *In particular,* $T_{\max}(\eta) = \infty$.

Remark: $\mathcal{O}(\sqrt{\mu})$ is emergent pattern size due to Ginzburg-Landau theory.

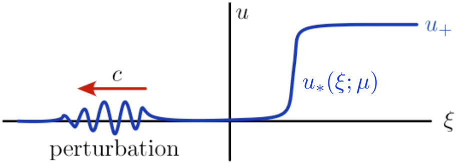
Proof This also has two steps:

- Show behavior of perturbations is governed by that at $-\infty$.
- Control behavior at $-\infty$ using mode filters.



Step 2a: behavior at $-\infty$

Strategy: linearize at rest state 0 and show nothing else matters.



Recall:

$$\mathcal{L}_- = D\partial_\xi^2 + c\partial_\xi + f_u(0)$$

Write

$$v_t = \mathcal{L}_- v + \mathcal{N}_-(v) + \Delta(p, v)$$

$\Delta(p, v) =$ difference in (non)linearization about u_* and about 0

Lemma For all $0 \le t \le T_{\max}(\eta)$,

$$\|\Delta(p, v)(t)\|_{H^1_{\mathrm{ul}}} \le C(\eta, \Lambda)\|w(t)\|_{H^1_{\mathrm{ul}}} \le C(\eta, \Lambda)\mathrm{e}^{-\Lambda t}\|w(0)\|_{H^1_{\mathrm{ul}}}.$$

Step 2b: control emergent pattern via mode filters

$$v_t = D\partial_x^2 v + f(v;\mu) + \underbrace{\Delta(p,v)}_{\mathcal{O}(e^{-\Lambda t})}$$

Ginzburg-Landau formalism for

$$v_t = D\partial_x^2 v + f(v;\mu) \tag{RD}$$

Consider modulated waves of the form

$$v(x,t) = \delta \mathrm{e}^{\mathrm{i}k_0 x + \mathrm{i}\omega_0 t} A(\delta x, \delta^2 t) e(k_0) + \mathrm{c.c.} \tag{MW}$$

Dynamics of amplitude $A(\delta x, \delta^2 t) = A(X, T)$

$$A_T = \lambda_2(0)\partial_X^2 A + \frac{\mu \lambda_0'(0)}{\delta^2} A - b|A|^2 A. \tag{GL}$$

To show (GL) really controls the behavior of v, we need:

- Approximation: Given solution of (GL) and (MW), $\exists$ nearby sol'n of (RD)
- Attractivity: Given sol'n of (RD), $\exists$ a nearby (MW) via (GL)

Note: Both have been shown by Schneider and Mielke for (RD), ie $\Delta = 0$

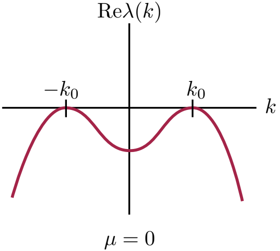
Step 2b: define mode filters

Linear operator at $-\infty$, where pattern will form:

$$\mathcal{L}_-(\partial_x) = D\partial_x^2 + f_u(0), \qquad \hat{\mathcal{L}}_-(\mathrm{i}k) = -k^2 D + f_u(0)$$

Turing bifurcation: for any $\mu \approx 0$ and $k \approx k_0 \neq 0$, near critical mode

$$\lambda(k)\hat{\mathcal{L}}_-(\mathrm{i}k) = \lambda(k)\hat{e}(k), \qquad \hat{\mathcal{L}}_-^*(\mathrm{i}k), \qquad \langle \hat{e}(k), \hat{e}^*(k)\rangle = 1$$



Mode filter: defined in Fourier space

$$(\widehat{P}_{\mathrm{mf}}^{\pm}\hat{u})(k) = \hat{\chi}(2(k \mp k_0))\langle \hat{e}^*(k,\mu), \hat{u}(k)\rangle \hat{e}(k,\mu), \qquad \widehat{P}_{\mathrm{mf}}^{\mathrm{c}} = \widehat{P}_{\mathrm{mf}}^{+} + \widehat{P}_{\mathrm{mf}}^{-}$$

Note: doesn't yield projections!

$$P_{\mathrm{mf}}^{\mathrm{c}} \circ P_{\mathrm{mf}}^{\mathrm{c}} \neq P_{\mathrm{mf}}^{\mathrm{c}}$$

Step 2b: new Ansatz via mode filters

Standard Ansatz: (Turing: $k_0 \neq 0$, $\omega_0 = 0$)

$$v(x,t) = \delta \mathrm{e}^{\mathrm{i}k_0 x} A(\delta x, \delta^2 t) e(k_0) + \mathrm{c.c.}$$

Tells us $v = v(A)$. For attractivity, need better "guess" and other direction!

$$\begin{aligned} v(x;A) &= \delta \mathrm{e}^{\mathrm{i}k_0 x} \mathcal{F}^{-1}\left[\hat{\chi}(k)\hat{e}(k+k_0)\mathcal{F}(A(\delta x))\right] + \mathrm{c.c.} \\ A(X;v) &= \frac{1}{\delta}\mathrm{e}^{-\mathrm{i}k_0 X/\delta}(p^{+}_{\mathrm{mf}} v)(X/\delta) \end{aligned}$$

Extract critical modes from v to get A.

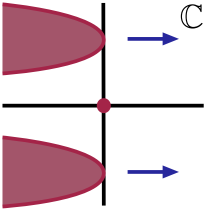
This allows us to:

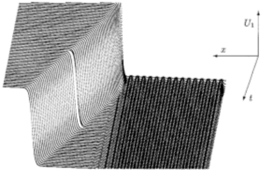
- Prove both attractivity and approximation, for (mod RD)
- Therefore, v behaves as predicted by A
- Amplitude of pattern must saturate

$$\|v(\cdot,t)\|_{H^1_{\mathrm{ul}}} \leq K\left(\|v(\cdot,0)\|_{H^1_{\mathrm{ul}}} + \sqrt{|\mu|}\right)$$

Summary

Essential Hopf bifurcation caused by rest state behind front:





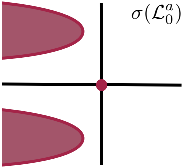
After bifurcation, intuitively:

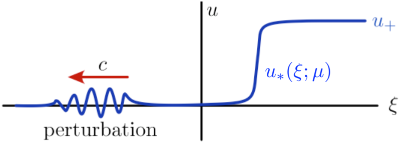
- Front becomes linearly unstable
- No other nearby solutions exist [Sandstede & Scheel 01]
- Numerically: front outruns perturbation
- If growth of perturbation saturates, front should be nonlinearly stable

Summary

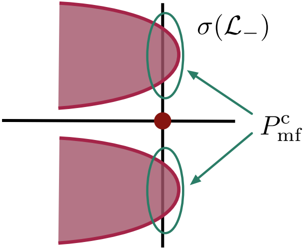
Proof had two steps:

1) Show decay in exponentially weighted space if pattern growth saturates





2) Show, via mode filters, that growth does indeed saturate



$$v_t = D\partial_x^2 v + f(v;\mu) + \mathcal{O}(e^{-\Lambda t}) \qquad \text{(RD)}$$

$$A_T = \lambda_2(0)\partial_X^2 A + \frac{\mu\lambda_0'(0)}{\delta^2} A - b|A|^2 A \qquad \text{(GL)}$$