Spectral Stability and the Maslov Index

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Motivation: Sturm-Liouville Theory

Eigenvalue problem:

$$\lambda u = u_{xx} + \alpha(x)u = \mathcal{L}u, \qquad x \in (a, b)$$
$$u(a) = u(b) = 0$$

Prüfer coordinates: define (r, θ) via

$$u = r\sin\theta, \qquad u_x = r\cos\theta$$

To obtain

$$\begin{aligned} r_x &= r(1 + \lambda - \alpha(x))\cos\theta\sin\theta \\ \theta_x &= \cos^2\theta + (\alpha(x) - \lambda)\sin^2\theta \end{aligned}$$

Observe:

- θ equation decouples
- $\{r = 0\}$ is invariant, so for a nontrivial solution,

$$u(x;\lambda) = 0 \qquad \text{if and only if} \qquad \theta(x;\lambda) = j\pi, \quad j \in \mathbb{Z}$$

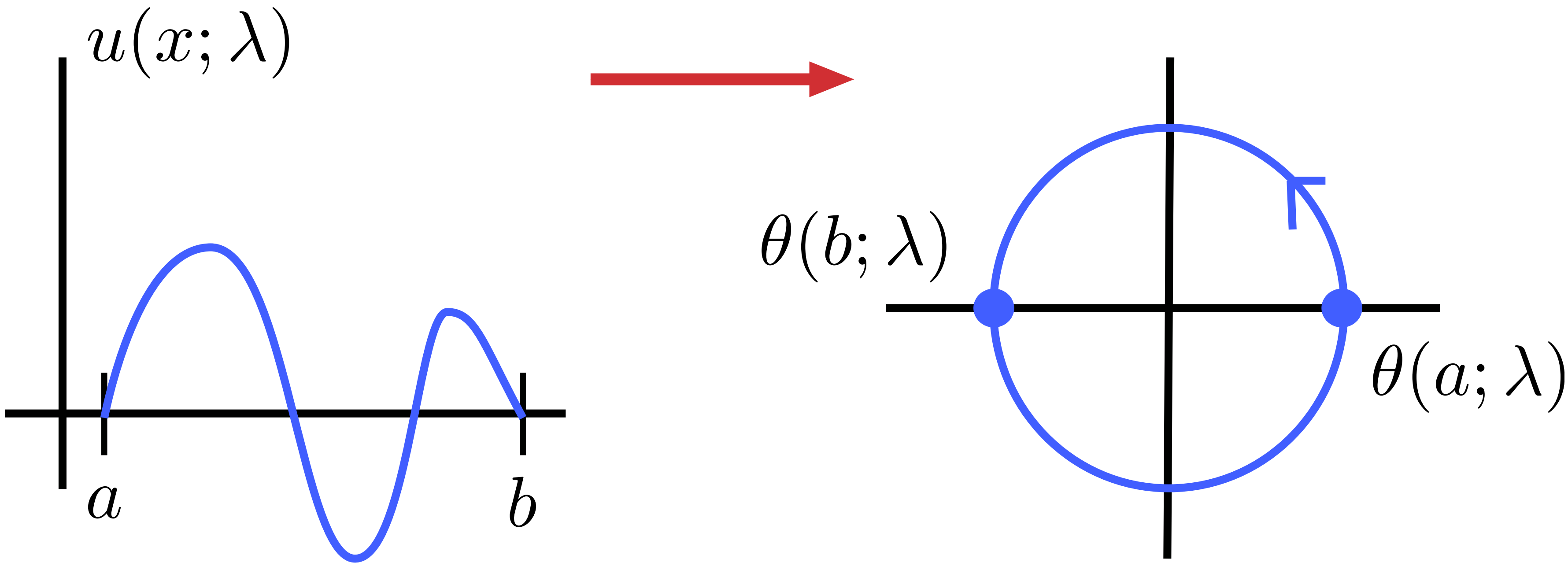
- For $\lambda \ll -1$, $\theta' > 0$, so solutions will be forced to oscillate

Motivation: Sturm-Liouville Theory

Focus on the angular dynamics:

$$\theta_x = \cos^2\theta + (\alpha(x) - \lambda)\sin^2\theta$$

Let $\theta(a;\lambda) = 0$ be the "initial condition" and evolve in x.
If $\theta(b;\lambda) \in \{j\pi\}$, then λ is an eigenvalue.

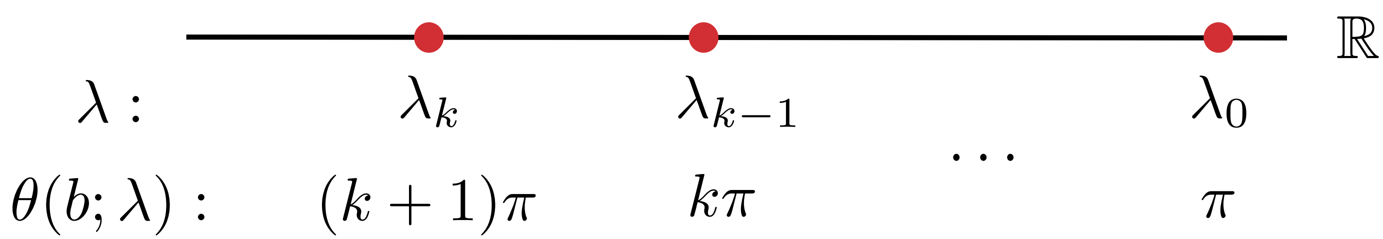


Motivation: Sturm-Liouville Theory

Angular dynamics:

$$\theta_x = \cos^2\theta + (\alpha(x) - \lambda)\sin^2\theta, \qquad x \in (a, b)$$

- Initial condition: $\theta(a;\lambda) = 0$; flow forward and see if $\theta(b;\lambda) \in \{j\pi\}$
- For some $\lambda \ll -1$ there must be an eigenvalue. Fix such a λ_k: $\theta(b;\lambda_k) = (k+1)\pi$.
- Increase λ until you again land in $\{j\pi\}$, which is the eigenvalue λ_{k-1}.



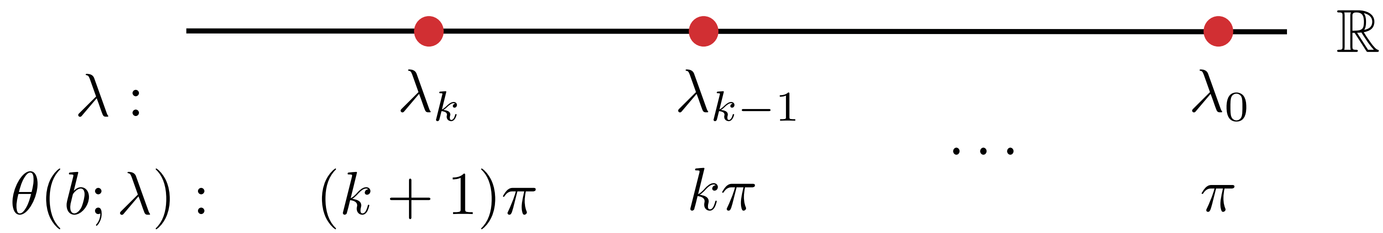
- Process stops at largest λ_0; θ no longer can complete one half-rotation

Motivation: Sturm-Liouville Theory

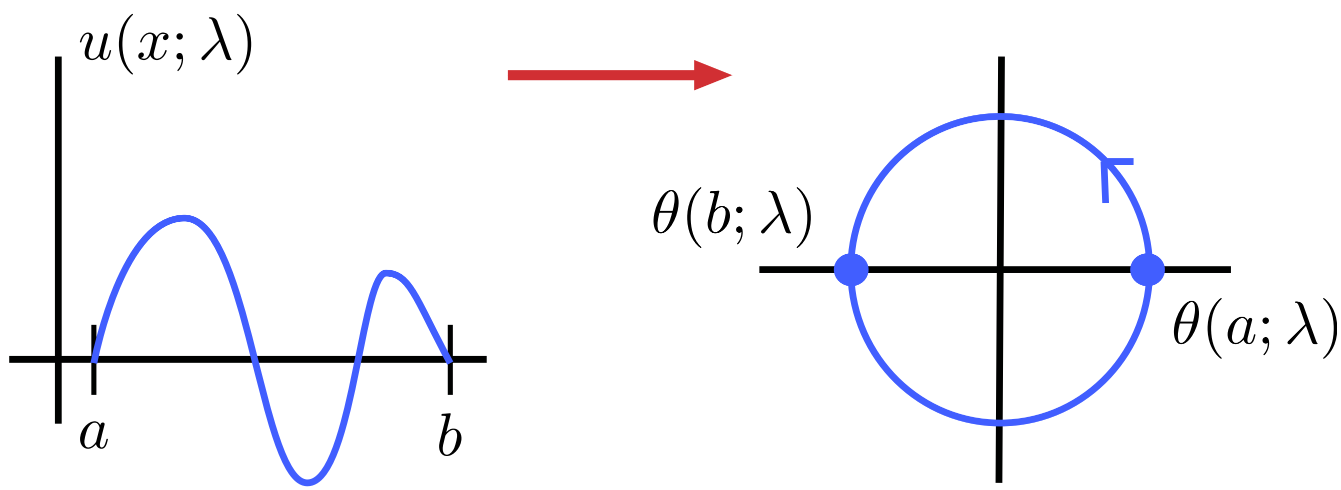
Using these ideas one can show:

$$\lambda u = u_{xx} + \alpha(x)u = \mathcal{L}u, \qquad x \in (a, b)$$
$$u(a) = u(b) = 0$$

- There exists a decreasing sequence of simple eigenvalues $\lambda_0 > \lambda_1 > \dots$



- Corresponding eigenfunctions $u_k(x)$ have k simple zeros in (a, b)



$u_2(x;\lambda_2)$ $\qquad$ $\theta(b;\lambda_2) = 3\pi$

Motivation: Sturm-Liouville Theory

Consequences for stability in scalar reaction-diffusion equations:

$$u_t = u_{xx} + f(u), \qquad x \in \mathbb{R}$$

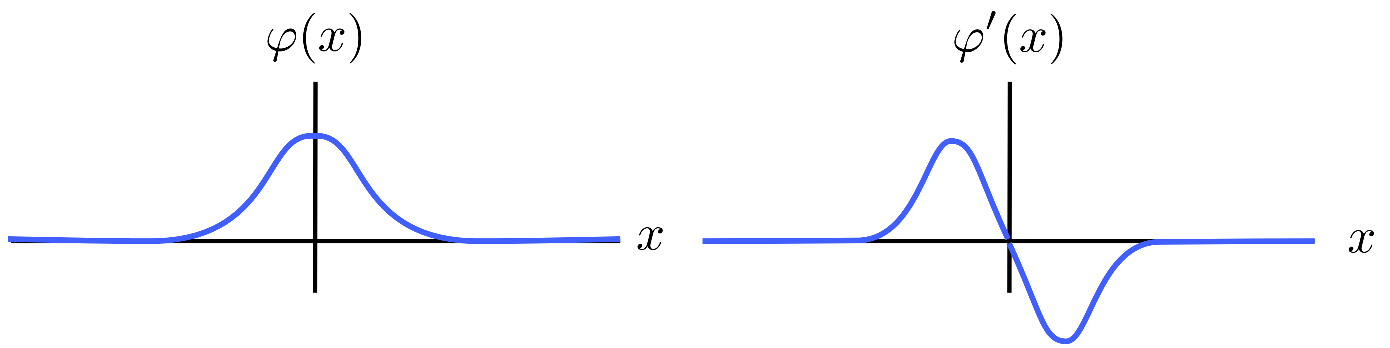
Linearize about stationary solution $\varphi(x)$:

$$\lambda u = u_{xx} + f'(\varphi(x))u = \mathcal{L}u, \qquad x \in \mathbb{R}, \qquad u \in L^2(\mathbb{R})$$

Notice:

$$0 = \varphi_{xx} + f(\varphi(x)) \qquad \Rightarrow \qquad 0 = (\varphi_x)_{xx} + f'(\varphi(x))\varphi_x = \mathcal{L}\varphi_x$$

Immediately conclude any pulse must be unstable:



$$\varphi_x = u_1 \qquad \lambda_1 = 0 \qquad \Rightarrow \qquad \lambda_0 > 0$$

We are effectively using the zeros as a proxy for the eigenvalues!

Motivation: Sturm-Liouville Theory

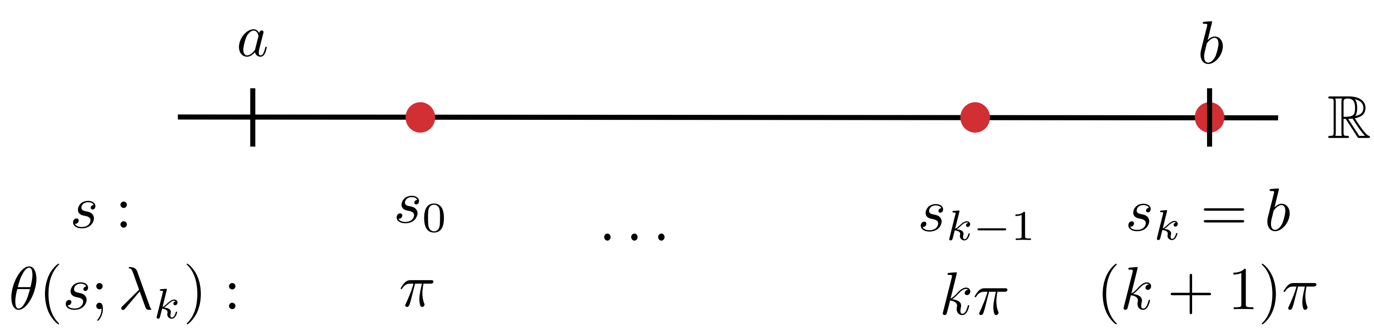
Related concept of conjugate points:

$$\theta' = \cos^2\theta + (\alpha(x) - \lambda)\sin^2\theta,$$

Instead of fixing the domain and varying λ, now fix λ and vary the domain:

$$x \in (a, s), \qquad s \in (a, b] \quad \text{is a parameter}$$

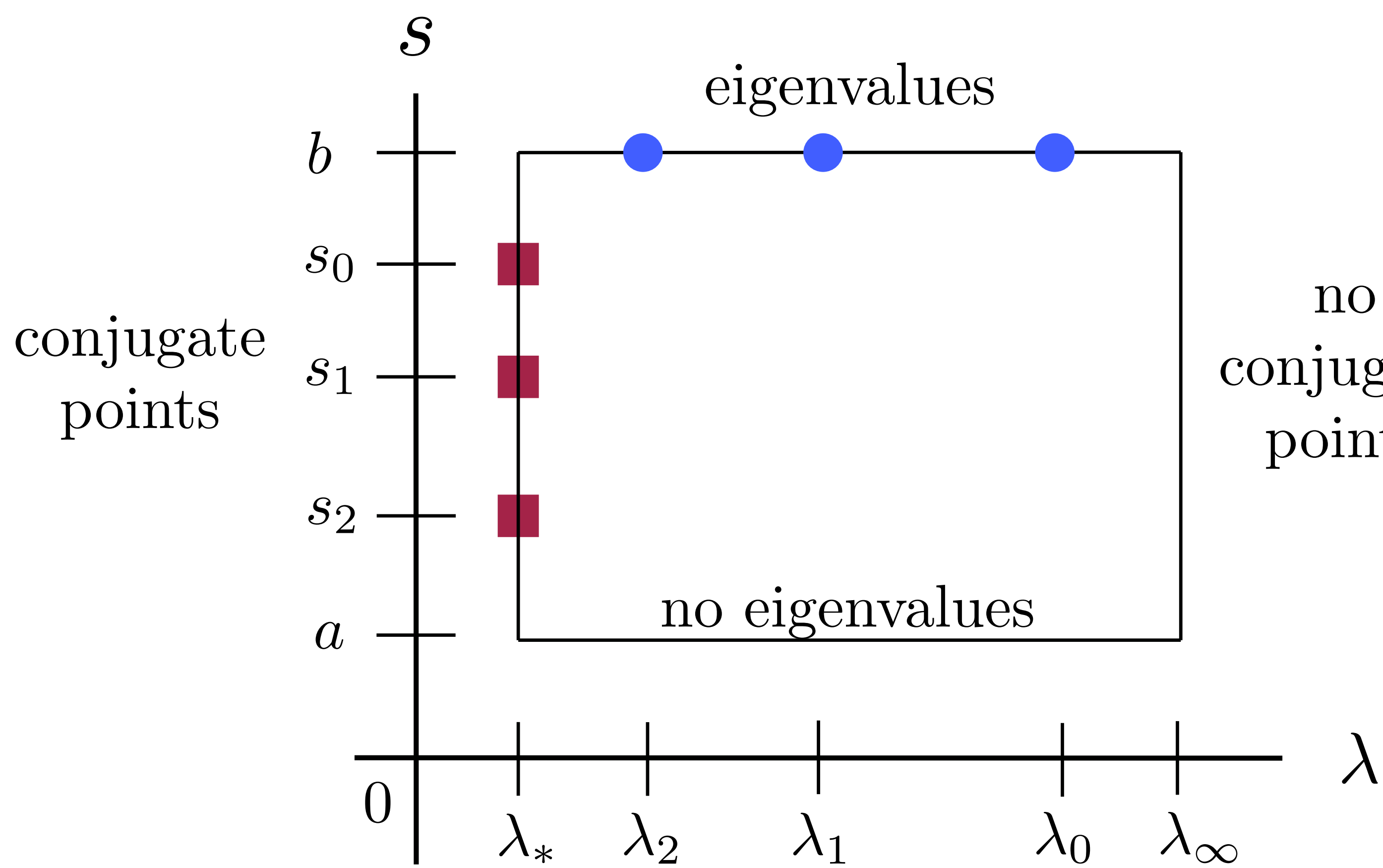
- Initial condition: $\theta(a; \lambda) = 0$; flow forward and see if $\theta(s; \lambda) \in \{j\pi\}$
- Fix $\lambda = \lambda_k$ to be an eigenvalue, so if $s = b$ we know $\theta(b; \lambda_k) = (k+1)\pi$
- Decrease s until you again land in $\{j\pi\}$, which is the conjugate point s_{k-1}.



- Process stops at largest s_0; θ no longer can complete one half-rotation

Motivation: Sturm-Liouville Theory

"Square": Relationship between eigenvalues and conjugate points:



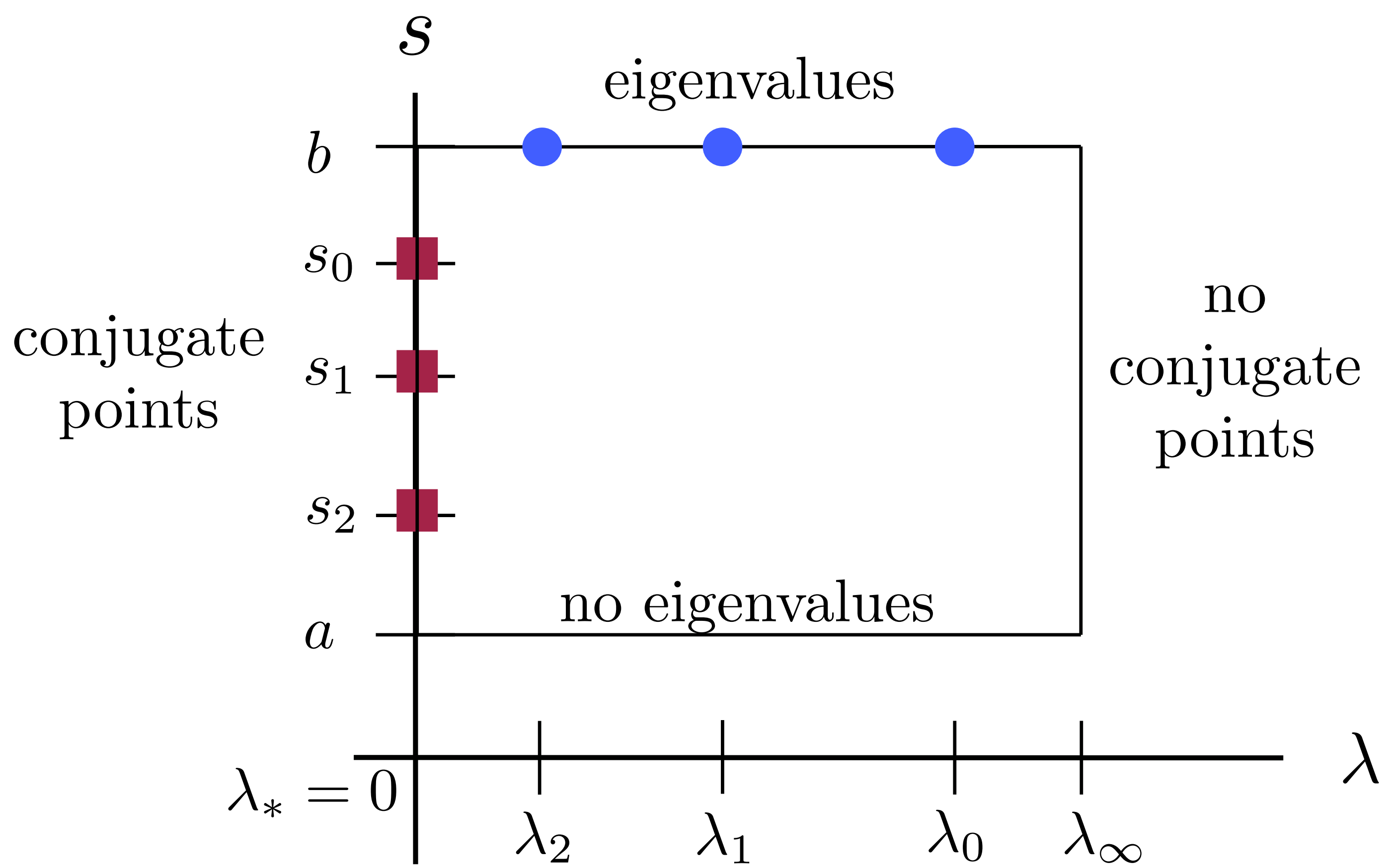
$\#\{\text{conjugate points for } \lambda = \lambda_*\} = \#\{\text{eigenvalues } \lambda > \lambda_*\}$

One can also prove:

- No eigenvalues for $s = a$; no "time" to oscillate.
- No conjugate points for $\lambda = \lambda_\infty$ large; ODE or spectral analysis.

Motivation: Sturm-Liouville Theory

To analyze stability, choose $\lambda_* = 0$:



Number of conjugate points = number of unstable eigenvalues = Morse($\mathcal{L}$)

This is a simple case of what is often called the Morse Index Theorem, and it goes back to the work of Morse, Bott, etc, in the 50s.

Motivation: Sturm-Liouville Theory

Summary so far:

- Sturm-Liouville theory, when it applies, is powerful: in scalar reaction-diffusion equations, pulses are unstable; one needs no details about the equation or underlying wave.
- In general, finding eigenvalues can be hard; sometimes finding conjugate points is easier: eg count zeros of φ_x.
- In the scalar case, conjugate points can be analyzed via the winding of a phase; monotonicity in λ and s was key.

Can we generalize this to systems?

$$u \in \mathbb{R}^n$$

or multidimensional domains?

$$x \in \Omega \subset \mathbb{R}^d$$

Systems in one spatial dimension

$$u_t = u_{xx} + f(u), \qquad x \in \mathbb{R}, \qquad u \in \mathbb{R}^n$$

Key restrictive assumption:

$$f(u) = \nabla G(u), \qquad G : \mathbb{R}^n \to \mathbb{R}$$

Will imply linearized operator is self-adjoint and provide a symplectic structure.

Stationary solution $\varphi(x)$; suppose it is a pulse:

$$\lim_{x \to \pm\infty} \varphi(x) = \varphi_\infty$$

Eigenvalue equation:

$$\lambda u = u_{xx} + \nabla^2 G(\varphi(x))u = \mathcal{L}u, \qquad u \in L^2(\mathbb{R}, \mathbb{R}^n)$$

Natural assumption: the essential spectrum of $\mathcal{L}$ is stable. Equivalently,

$$\nabla^2 G(\varphi_\infty) < 0.$$

Systems in one spatial dimension

Eigenvalue equation:

$$\lambda u = u_{xx} + \nabla^2 G(\varphi(x))u = \mathcal{L}u, \qquad u \in \mathbb{R}^n, \qquad x \in \mathbb{R}$$

Write as a first-order system:

$$\begin{aligned}
\frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} 0 & I \\ (\lambda - \nabla^2 G(\varphi(x))) & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \qquad\qquad \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^{2n} \\
&= \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} (\lambda - \nabla^2 G(\varphi(x))) & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \\
&= J\mathcal{B}(x;\lambda)\begin{pmatrix} u \\ v \end{pmatrix}
\end{aligned}$$

Note that λ can be taken to be real and $\mathcal{B}(x;\lambda)$ is a symmetric matrix.

Can we develop a Sturm-Liouville-like theory for such eigenvalue problems?

- Arnol'd (1967, 1985) generalized the notion of phase to $\mathbb{R}^n$ via the Maslov index and proved oscillation theorems.
- Can we connect his theory with eigenvalues?

Note: the above perspective is often called 'spatial dynamics'.

Oscillations in $\mathbb{R}^n$

Symplectic form:

$$\omega(U, V) = \langle U, JV \rangle_{\mathbb{R}^{2n}}.$$

Lagrangian-Grassmanian:

$$\Lambda(n) = \{\ell \subset \mathbb{R}^{2n} : \dim(\ell) = n, \quad \omega(U, V) = 0 \quad \forall U, V \in \ell\}.$$

Each Lagrangian plane has an associated frame matrix: $A, B \in \mathbb{R}^{n \times n}$

$$\ell = \left\{ \begin{pmatrix} A \\ B \end{pmatrix} u : u \in \mathbb{R}^n \right\} \qquad\qquad \ell \Leftrightarrow \begin{pmatrix} A \\ B \end{pmatrix}$$

Suppose we have a path of Lagrangian planes,

$$\ell(t) \in \Lambda(n), \qquad t \in [a, b]$$

and we are interested in its intersections with a fixed reference plane, such as the Dirichlet plane:

$$\mathcal{D} = \left\{ U = \begin{pmatrix} 0 \\ v \end{pmatrix} : v \in \mathbb{R}^n \right\} \in \Lambda(n)$$

This is very similar to looking for conjugate points.

Oscillations in $\mathbb{R}^n$

Associate the path $\ell(t)$ with a frame matrix

$$\ell(t) \Leftrightarrow \begin{pmatrix} A(t) \\ B(t) \end{pmatrix}$$

There is a well-defined angle $\phi(t)$ satisfying

$$e^{\mathrm{i}\phi(t)} = \det[\underbrace{(A(t) - \mathrm{i}B(t))(A(t) + \mathrm{i}B(t))^{-1}}_{=:W(t)}],$$

which utilizes the fact that $W(t)$ is unitary, hence has spectrum on $\mathbb{S}^1$. Also,

$$\dim[\ker(W(t) + I)] = \dim(\ell(t) \cap \mathcal{D})$$

The Maslov index counts, with multiplicity and direction, the number of times an eigenvalue of $W(t)$ crosses through -1. It is related to the fact that

$$\pi_1(\Lambda(n)) = \mathbb{Z}.$$

If $\ell(t)$ is a loop, its Maslov index is just its equivalence class in $\pi_1(\Lambda(n))$.

[Arnol'd 1967, Furutani 2004, and Howard, Latushkin, Sukhtayev 2017].

Systems in one spatial dimension

Back to our eigenvalue problem:

$$\frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = J\mathcal{B}(x;\lambda)\begin{pmatrix} u \\ v \end{pmatrix}$$

The assumption $\nabla^2 G(\varphi_\infty) < 0$ implies

$$J\mathcal{B}(\pm\infty;\lambda) = \begin{pmatrix} 0 & I \\ (\lambda - \nabla^2 G(\varphi_\infty)) & 0 \end{pmatrix}$$

is hyperbolic with stable/unstable eigenspace of dimension n. Therefore,

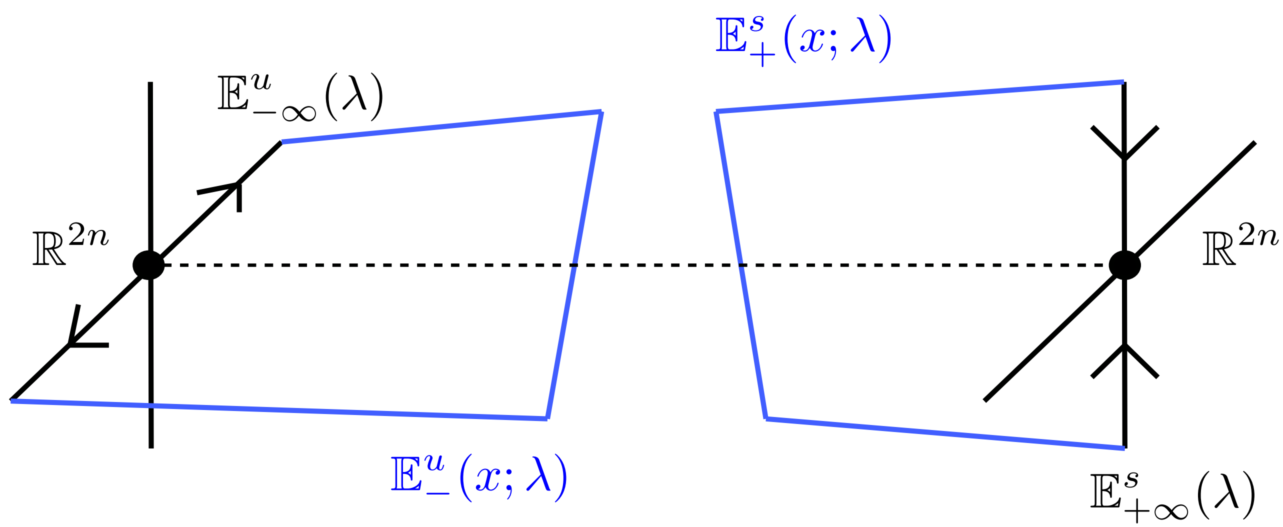
$$\dim(\mathbb{E}^{s,u}_{\pm}(x;\lambda)) = n$$

In other words, the subspaces of solutions asymptotic to these stable/unstable eigenspaces at $\pm\infty$ must also have dimension n.

Systems in one spatial dimension

$$\frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = J\mathcal{B}(x;\lambda)\begin{pmatrix} u \\ v \end{pmatrix}$$

For an eigenfunction, we need $(u, v)(x; \lambda) \in \mathbb{E}^u_-(x;\lambda) \cap \mathbb{E}^s_+(x;\lambda)$:



It turns out these are both paths of Lagrangian subspaces!

Alternatively, look for conjugate points:

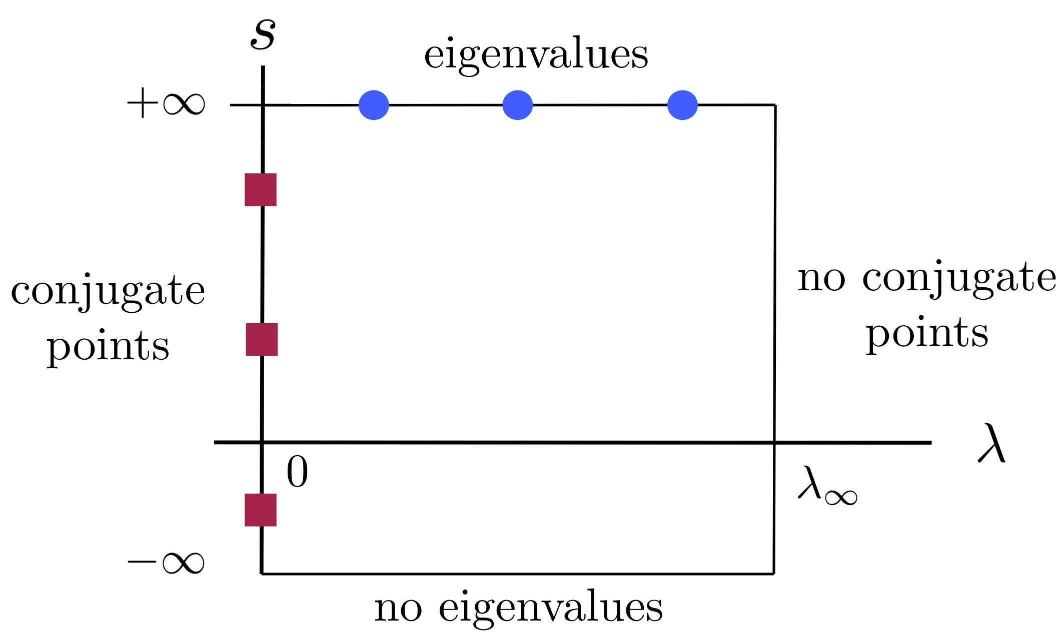
$\ell(x;\lambda) = \mathbb{E}^u_-(x;\lambda) \in \Lambda(n)$ When is $\mathbb{E}^u_-(x;\lambda) \cap \mathcal{D} \neq \{0\}$?

Systems in one spatial dimension

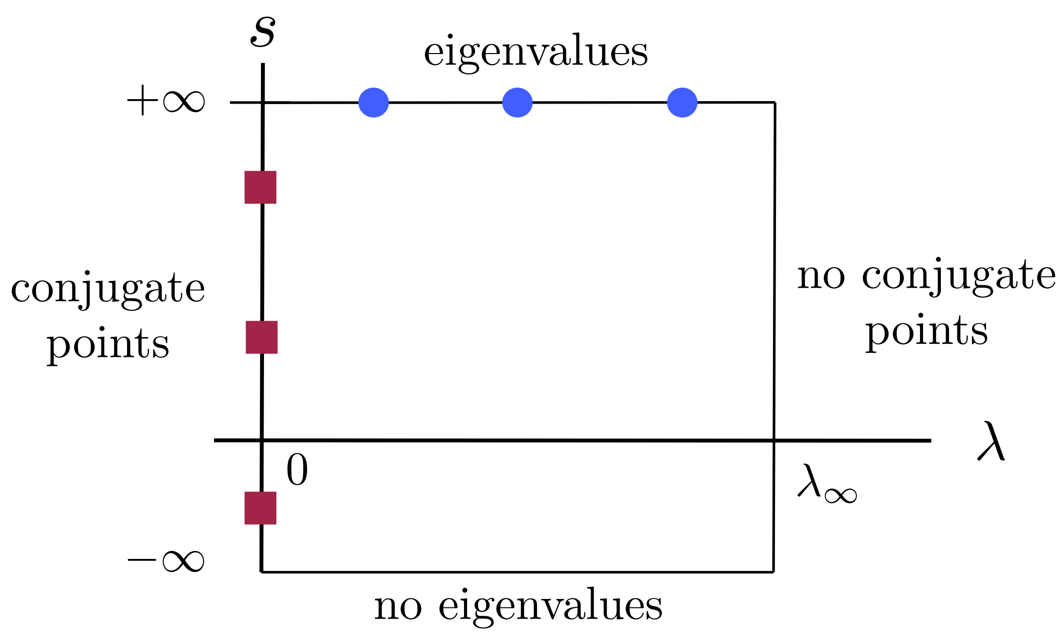
Main results of [B., Cox, Jones, Latushkin, McQuighan, Suhktayev '18]:

- Proved "square" relating eigenvalues to conjugate points.
- Proved a pulse solution is necessarily unstable, as in the scalar case.

$$\lambda u = u_{xx} + \nabla^2 G(\varphi(x))u = \mathcal{L}u, \qquad u \in \mathbb{R}^n, \qquad \mathrm{dom}(\mathcal{L}) = H^2(\mathbb{R}) \subset L^2(\mathbb{R})$$



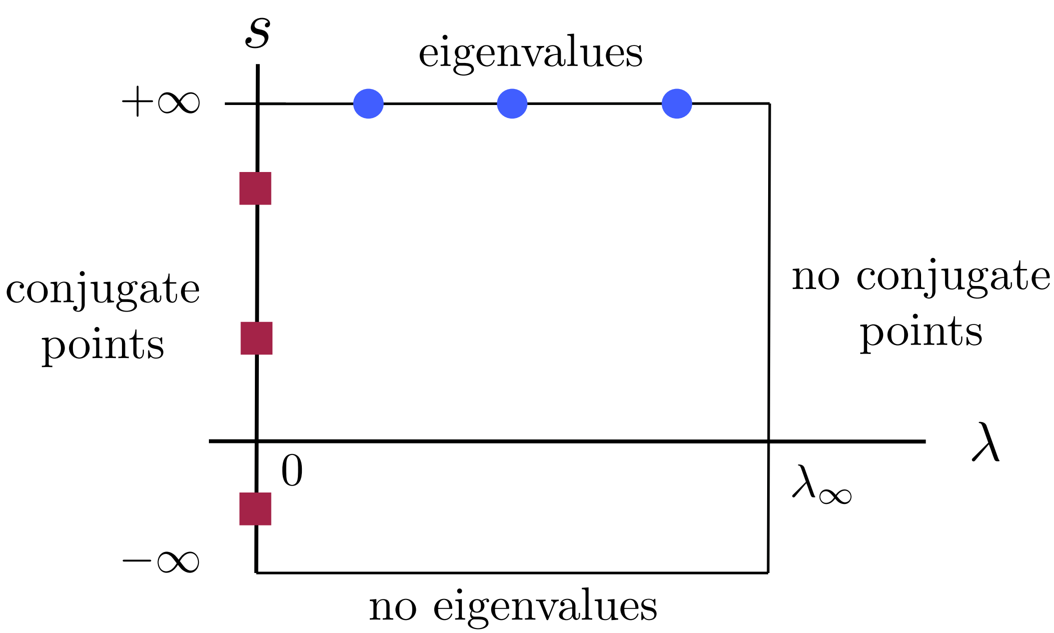
Systems in one spatial dimension



Some ideas in proof:

- Compactify domain $s \in (-\infty, \infty) \to \tilde{s} \in [-1, 1]$
- Path of Lagrangian subspaces $\ell(\tilde{s}; \lambda) = \mathbb{E}^u_-(\tilde{s}; \lambda)$ around entire square
- Since $[-1, 1] \times [0, \lambda_\infty] \subset \mathbb{R}^2$ is a trivial loop, the Maslov index of $\ell(\tilde{s}; \lambda)$ around it must be zero.
- Show bottom and right side have no intersections.
- Prove monotonicity: determine sign of crossings on top $(-)$ and left $(+)$.
- Use "crossing form" to characterize Maslov index [Robbin, Salamon 1993]

Systems in one spatial dimension



Additional comments

- Pulse instability follows from a symmetry argument; use reversibility to prove there must be at least one conjugate point.
- There are many (recent) theoretical results using the Maslov index to relate unstable eigenvalues to conjugate points, but there are very few applications of these results to actually determine (in)stabilily.
- Current work with J. Jaquette: use validated numerics to count conjugate points; much faster than validated Evans function computations.

Multiple space dimensions

Eigenvalue problem:

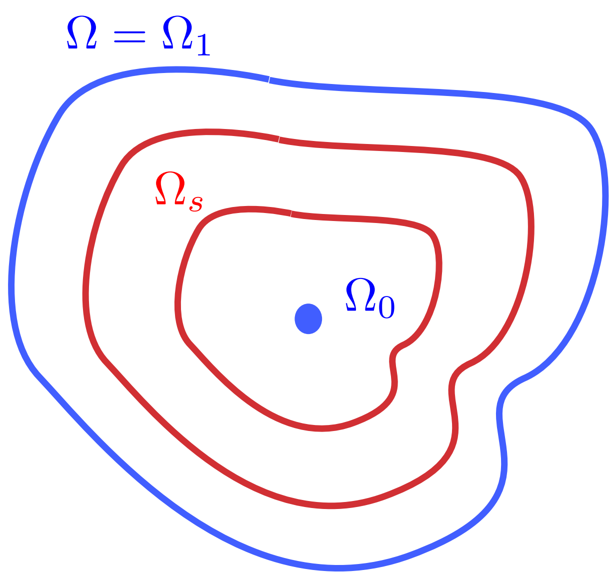
$$\mathcal{L}u = \Delta u + V(x)u = \lambda u, \qquad x \in \Omega \subset \mathbb{R}^d, \qquad u \in \mathbb{R}, \qquad \lambda \in \mathbb{R}$$
$$u|_{\partial\Omega} = 0$$

Can we develop anything like Sturm-Liouville Theory here?

- Notion of conjugate points?
- Lagrangian structure and Maslov index?

Family of domains [Smale 65]:

$$\{\Omega_s : 0 \le s \le 1\}, \qquad \Omega_1 = \Omega, \qquad \Omega_0 = \{x_0\}.$$



Multiple space dimensions

Path of subspaces:

$$\ell(s;\lambda) = \left\{ \left(u, \frac{\partial u}{\partial n} \right) \Big|_{\partial\Omega_s} : u \in H^1(\Omega_s), \quad \Delta u + V(x)u = \lambda u, \quad x \in \Omega_s \right\}$$

subspace of solutions on Ω_s with no reference to boundary conditions

Dirichlet subspace:

$$\mathcal{D} = \left\{ \left(u, \frac{\partial u}{\partial n} \right) \Big|_{\partial\Omega} = \left(0, \frac{\partial u}{\partial n} \right) \Big|_{\partial\Omega} : u \in H^1(\Omega_s) \right\}$$

reference plane determined by boundary conditions

Conjugate point:

$$\ell(s;\lambda) \cap \mathcal{D} \neq \{0\}$$

Hilbert space

$$\mathcal{H} = H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega), \qquad \omega((f_1, g_1), (f_2, g_2)) = \langle g_2, f_1 \rangle - \langle g_1, f_2 \rangle$$

Both $\ell(s;\lambda)$ and $\mathcal{D}$ are in the Fredholm-Lagrangian Grassmannian of $\mathcal{H}$. [Deng, Jones '11], [Cox, Jones, Latushkin, Suhktayev '16], ... relate conjugate points to eigenvalues via the Maslov index; more general BCs and systems.

Spatial dynamics in $\mathbb{R}^d$?

Does this suggest a "spatial dynamics" for $\mathbb{R}^d$? Consider

$$0 = \Delta u + F(x, u), \qquad x \in \Omega \subset \mathbb{R}^d$$

Family of domains parameterized by family of diffeomorphisms:

$$\psi_s : \Omega \to \Omega_s, \qquad s \in [0, 1], \qquad \Omega_1 = \Omega, \qquad \Omega_0 = \{x_0\}.$$

Define boundary data via

$$f(s; y) = u(\psi_s(y)), \qquad g(s; y) = \frac{\partial u}{\partial n}(\psi_s(y)), \qquad s \in [0, 1], \qquad y \in \partial\Omega$$

and trace map

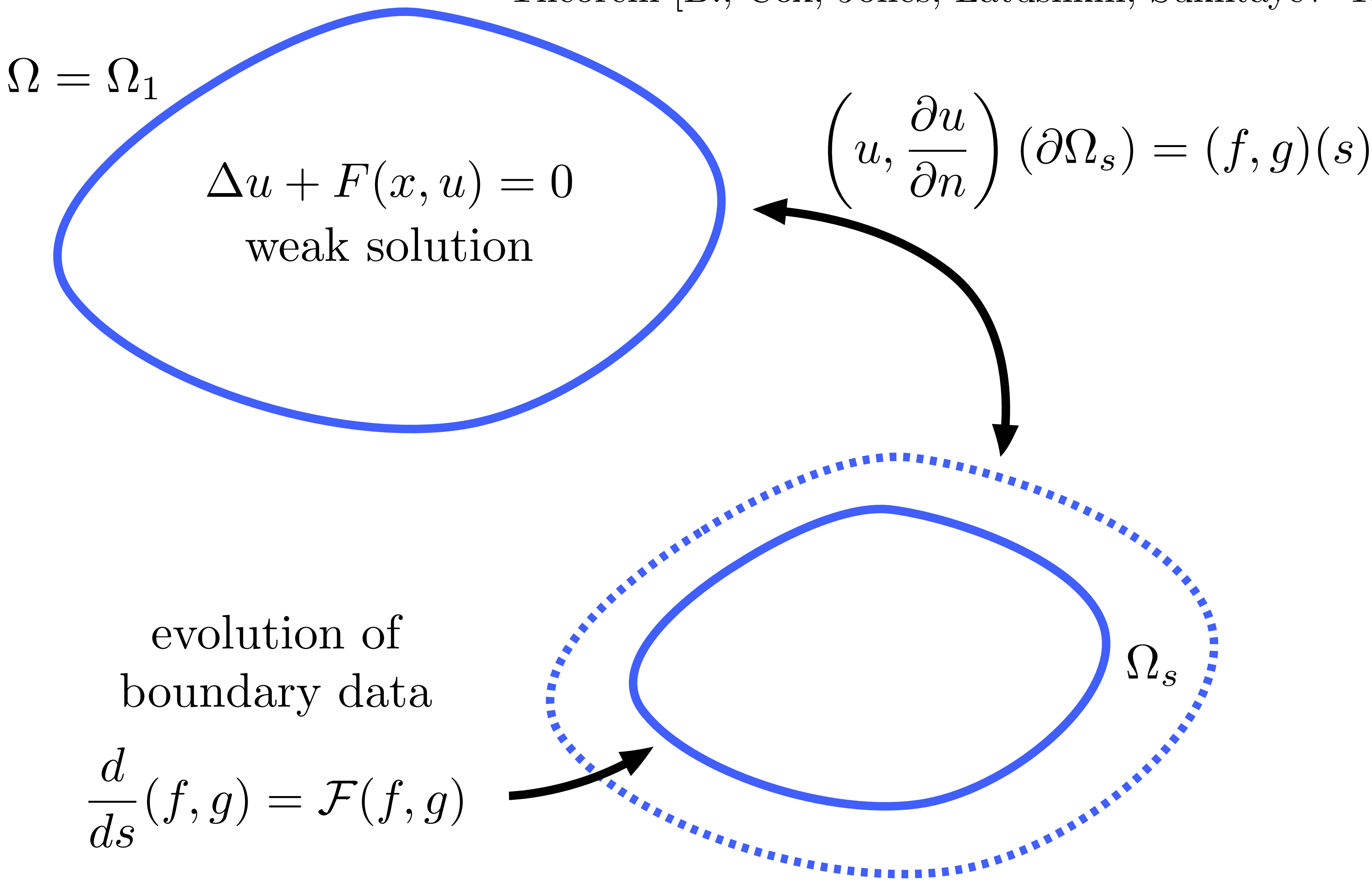
$$\mathrm{Tr}_s u = (f(s), g(s)).$$

Obtain an equivalent first-order system

$$\frac{d}{ds}\begin{pmatrix} f \\ g \end{pmatrix} = \mathcal{F}(f, g)$$

Spatial dynamics in $\mathbb{R}^d$?

Theorem [B., Cox, Jones, Latushkin, Sukhtayev '19]:



Spatial dynamics in $\mathbb{R}^d$?

So, rather than solving $\Delta u + F(x,u) = 0$, we can instead solve

$$\frac{d}{ds}\begin{pmatrix} f \\ g \end{pmatrix} = \mathcal{F}(f,g),$$

a somewhat technical and not-fun-to-look-at equation. What have we gained?

Consider $\Omega = \mathbb{R}^d$. We can choose to shrink the domain using spheres:

$$\Omega_s = \{x \in \mathbb{R}^d : |x| < s\}, \qquad s \in (0,\infty)$$

In terms of the polar coordinates $(r,\theta) \in (0,\infty) \times \mathbb{S}^{d-1}$,

$$\Delta u = u_{rr} + \frac{n-1}{r} u_r + \frac{1}{r^2} \Delta_{\mathbb{S}^{d-1}} u,$$

and our equation is now not so bad:

$$\frac{d}{ds}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -s^{-2}\Delta_{\mathbb{S}^{d-1}} & -(d-1)s^{-1} \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} + \begin{pmatrix} 0 \\ -F(t,\theta,s) \end{pmatrix}$$

Spatial dynamics in $\mathbb{R}^d$?

$$\frac{d}{ds}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -s^{-2}\Delta_{\mathbb{S}^{d-1}} & -(d-1)s^{-1} \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} + \begin{pmatrix} 0 \\ -F(s,\theta,f) \end{pmatrix}$$

- We have shown the linear part of this equation admits an exponential dichotomy (after rescaling time $s = e^{\tau}$).
- For $d = 3$, the dichotomy can be written explicitly in terms of the spherical harmonics of the Laplacian.
- This allows one to potentially construct solutions to the nonlinear equation that are not necessarily radially symmetric.
- We hope this will be a useful method for studying multidimensional waves and patterns.

[Ongoing work with Cox, Jones, Latushkin, and Sukhtayev]

Summary

Sturm-Liouville Theory allows one to connect eigenvalues with conjugate points for scalar, second order equations.

The Maslov index allows for a generalization of this to systems of equations in one space dimension, and also to multiple spatial dimensions.

Many of the results are abstract; main application so far is to prove pulses in reaction-diffusion systems with gradient nonlinearity are necessarily unstable. If you have any ideas about possible applications, please let me know!

The ideas used in the multidimensional case lead to a formulation of 'spatial dynamics' in multiple spatial dimensions.

THANK YOU!!!