An introduction to recent applications of the Maslov index in dynamical systems and PDEs

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Outline of talk

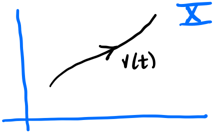
I'll give some background on how a topological invariant called the Maslov index has recently been utilized to understand stability in partial differential equations and dynamical systems. In particular, I will:

- Discuss the concept of stability in a dynamical systems context.
- Describe the topological invariant known as the Maslov index.
- Explain some connections between the Maslov index and stability.

I hope this will provide context for the talks that will follow.

Stability

$$\frac{\mathrm{d}}{\mathrm{d}t} v = F(v), \qquad v(t) \in X$$

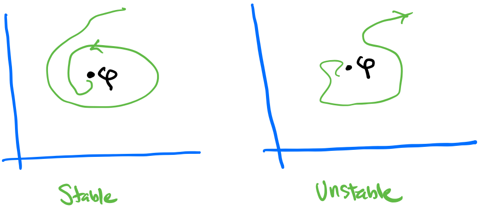


Examples:

- ODEs: $X = \mathbb{R}^n$, $F(v) = -v + v^2$
- PDEs: $X = L^2(\mathbb{R})$, $F(v) = \partial_x^2 v - v + v^2$

Stationary solution: $0 = F(\varphi)$

Stability (temporal): a stationary solution φ is stable if any solution that starts close to it converges to it.

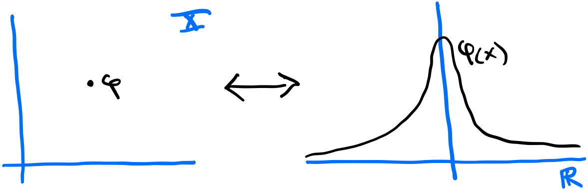


Why is stability important for understanding dynamics?

Ideal scenario: one understands the (in)stability of all stationary solutions, as well as more complicated invariant sets, and can thus piece together a global understanding of the dynamics. This rarely happens.

What we often settle for: understanding the (in)stability of a single stationary solution, which provides information about how likely we are to observe that solution, either numerically or in the real world. Even this can be hard, especially if your dynamical system is a PDE.

For PDEs, stationary solutions can depend on the spatial variable:



Determining stability

Start with a stationary solution:

$$v_t = F(v), \qquad 0 = F(\varphi)$$

Stability Ansatz: arbitrary solution starts close to φ

$$v(t) = \varphi + u(t), \qquad u(0) \text{ is small}$$

Plug into equation:

$$\begin{aligned} u_t &= F(\varphi + u) \\ &= F(\varphi) + dF(\varphi)u + N(u) \\ &= \underbrace{dF(\varphi)}_{\mathcal{L}}\, u + N(u). \end{aligned}$$

Characterization of Stability

$$\varphi \text{ is stable} \qquad \Leftrightarrow \qquad \text{solutions to } u_t = \mathcal{L}u + N(u) \text{ decay to zero}$$

Determining stability via decay to zero of perturbations

$$u_t = \mathcal{L}u + N(u)$$

Spectral stability: The spectrum of $\mathcal{L}$ lies strictly in the left half of $\mathbb{C}$. Are there any unstable eigenvalues? Often the hardest to prove.

Linear stability: The associated semigroup decays, eg $\|e^{\mathcal{L}t}\| \leq Ce^{-\lambda t}$. Spectral stability often implies linear stability via a spectral mapping theorem.

$$u_t = \mathcal{L}u \qquad \Rightarrow \qquad u(t) = e^{\mathcal{L}t}u(0)$$

Nonlinear stability: Solutions to the full nonlinear equation decay. Linear stability often implies nonlinear stability through a representation like

$$u(t) = e^{\mathcal{L}t}u(0) + \int_0^t e^{\mathcal{L}(t-s)}N(u(s))\mathrm{d}s.$$

The Maslov index

The Maslov index can be viewed as a generalization of the winding number in $\mathbb{C}$ to the Lagrangian Grassmannian.

Start with a symplectic form on $\mathbb{R}^{2n}$, such as

$$\omega(U,V) = \langle U, JV \rangle_{\mathbb{R}^{2n}}, \qquad J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

Define the Lagrangian Grassmannian to be n-planes where ω vanishes:

$$\Lambda(n) = \{\ell \subset \mathbb{R}^{2n} : \dim(\ell) = n, \quad \omega(U,V) = 0 \quad \forall U, V \in \ell\}.$$

Represent paths in $\Lambda(n)$ via frame matrices: for appropriate $A, B \in \mathbb{R}^{n \times n}$

$$\ell(t) = \left\{ \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} w : w \in \mathbb{R}^n \right\} \qquad\qquad \ell(t) \Leftrightarrow \begin{pmatrix} A(t) \\ B(t) \end{pmatrix}$$

The Maslov index

Associate path with angle $\phi(t)$ and unitary matrix $W(t)$:

$$\ell(t) \Leftrightarrow \begin{pmatrix} A(t) \\ B(t) \end{pmatrix}, \qquad \mathrm{e}^{\mathrm{i}\phi(t)} = \det[\underbrace{(A(t) - \mathrm{i}B(t))(A(t) + \mathrm{i}B(t))^{-1}}_{=:W(t)}],$$

The Maslov index counts eigenvalues of $W(t)$ that cross -1, with multiplicity and direction. It is related to the fact that

$$\pi_1(\Lambda(n)) = \mathbb{Z}.$$

If $\ell(t)$ is a loop, its Maslov index is just its equivalence class in $\pi_1(\Lambda(n))$.

This framework also detects intersections with fixed reference planes:

$$\mathcal{D} = \begin{pmatrix} 0 \\ I_n \end{pmatrix} \qquad \Rightarrow \qquad \dim[\ker(W(t) + I)] = \dim(\ell(t) \cap \mathcal{D})$$

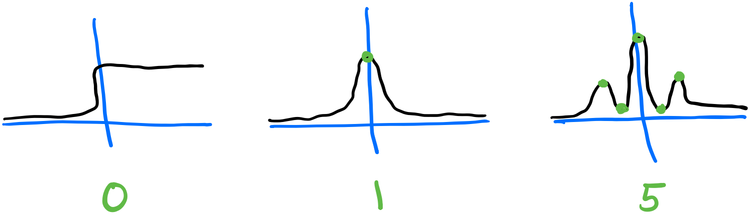
[Arnol'd 1967, Furutani 2004, and Howard, Latushkin, Sukhtayev 2017].

Connecting the Maslov index with stability

In certain situations, the Maslov index can be used to count the unstable eigenvalues of linear operators, hence providing information about (in)stability.

To understand this, we'll first look at a case study: Sturm-Liouville Theory.

The punchline: in certain scalar equations, one can count the unstable eigenvalues associated with φ by counting the number of local extrema of φ.



Case study: Sturm-Liouville Theory

Consider the equation

$$v_t = v_{xx} + f(v), \qquad v(x,t) \in \mathbb{R}, \qquad x \in [a,b], \qquad v \in L^2([a,b])$$

Take a stationary solution φ and linearize about it.

$$0 = \varphi_{xx} + f(\varphi) \qquad \Rightarrow \qquad \mathcal{L} = \partial_x^2 + f'(\varphi(x)).$$

Focus on spectral stability: look for positive eigenvalues of $\mathcal{L}$

$$\lambda u = \mathcal{L}u, \qquad x \in (a,b)$$
$$u(a) = u(b) = 0$$

Common technique: write this second order equation as a first order system:

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u \\ u_x \end{pmatrix} = \begin{pmatrix} u_x \\ [\lambda - f'(\varphi(x))]u \end{pmatrix}, \qquad \begin{pmatrix} u \\ u_x \end{pmatrix} \in \mathbb{R}^2$$

Our eigenvalue problem thus has a two-dimensional phase space.

Case study: Sturm-Liouville Theory

Turns out a change of variables is useful: define (r, θ) via

$$u = r \sin\theta, \qquad u_x = r\cos\theta$$

This gives

$$\begin{aligned} r_x &= r[1+\lambda - f'(\varphi(x))]\cos\theta\sin\theta \\ \theta_x &= \cos^2\theta - [\lambda - f'(\varphi(x))]\sin^2\theta \end{aligned}$$

Observe:

- θ equation decouples
- $\{r = 0\}$ is invariant, so for a nontrivial solution,

$$u(x;\lambda) = 0 \qquad \text{if and only if} \qquad \theta(x;\lambda) = j\pi, \quad j \in \mathbb{Z}$$

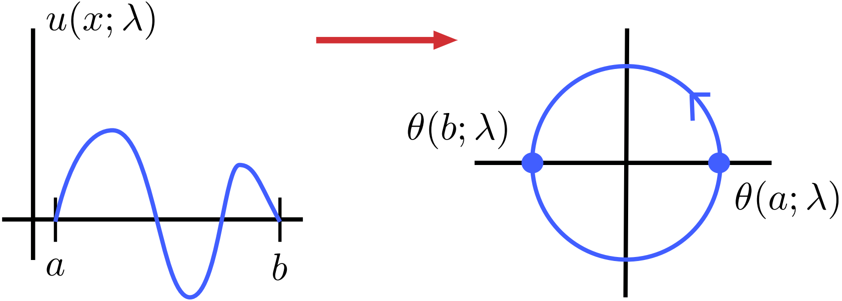
- For $\lambda \ll -1$, $\theta' > 0$, so solutions will be forced to oscillate

Case Study: Sturm-Liouville Theory

Focus on the angular dynamics:

$$\theta_x = \cos^2\theta - [\lambda - f'(\varphi(x))]\sin^2\theta$$

Let $\theta(a;\lambda) = 0$ be the "initial condition" and evolve in x.
If $\theta(b;\lambda) \in \{j\pi\}$, then λ is an eigenvalue.



Case Study: Sturm-Liouville Theory

Angular dynamics:

$$\theta_x = \cos^2\theta - [\lambda - f'(\varphi(x))]\sin^2\theta, \qquad x \in (a, b)$$

- Initial condition: $\theta(a;\lambda) = 0$; flow forward and see if $\theta(b;\lambda) \in \{j\pi\}$
- For some $\lambda \ll -1$ there must be an eigenvalue. Fix such a λ_k: $\theta(b;\lambda_k) = (k+1)\pi$.
- Increase λ until you again land in $\{j\pi\}$, which is the eigenvalue λ_{k-1}.

					$\mathbb{R}$
$\lambda:$	λ_k	λ_{k-1}	$\cdots$	λ_0	
$\theta(b;\lambda):$	$(k+1)\pi$	$k\pi$		π	

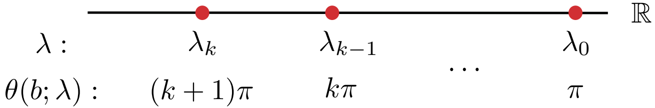
- Process stops at largest λ_0; θ no longer can complete one half-rotation.

Case Study: Sturm-Liouville Theory

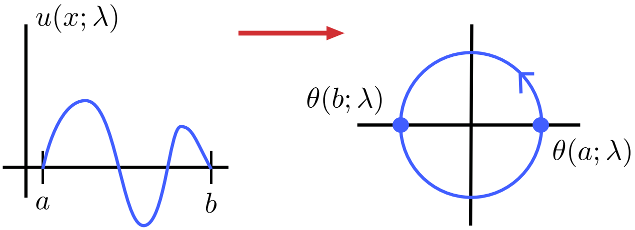
Using these ideas one can show:

$$\lambda u = \mathcal{L}u, \qquad x \in (a, b)$$
$$u(a) = u(b) = 0$$

- There exists a decreasing sequence of simple eigenvalues $\lambda_0 > \lambda_1 > \ldots$



- Corresponding eigenfunctions $u_k(x)$ have k simple zeros in (a, b)



$u_2(x; \lambda_2)$ $\qquad$ $\theta(b; \lambda_2) = 3\pi$

Case Study: Sturm-Liouville Theory

Consequences for stability in scalar reaction-diffusion equations:

$$v_t = v_{xx} + f(v), \qquad x \in \mathbb{R}$$

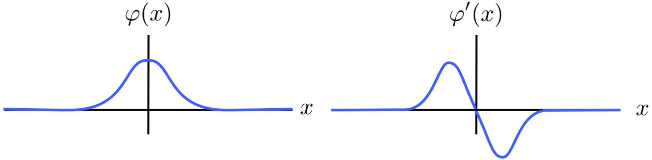
Linearize about stationary solution $\varphi(x)$:

$$\lambda u = u_{xx} + f'(\varphi(x))u = \mathcal{L}u, \qquad x \in \mathbb{R}, \qquad u \in L^2(\mathbb{R})$$

Notice:

$$0 = \varphi_{xx} + f(\varphi(x)) \qquad \Rightarrow \qquad 0 = (\varphi_x)_{xx} + f'(\varphi(x))\varphi_x = \mathcal{L}\varphi_x$$

Immediately conclude any pulse must be unstable:



$$\varphi_x = u_1 \qquad \lambda_1 = 0 \qquad \Rightarrow \qquad \lambda_0 > 0$$

We are effectively using the zeros of φ_x as a proxy for the eigenvalues!

Case Study: Sturm-Liouville Theory

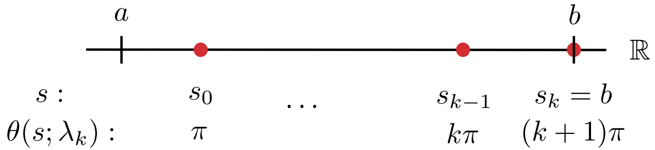
What does this have to do with the Maslov index?

$$\theta' = \cos^2\theta - [\lambda - f'(\varphi(x))]\sin^2\theta,$$

Instead of fixing the domain and varying λ, now fix λ and vary the domain:

$$x \in (a, s), \qquad s \in (a, b] \quad \text{is a parameter}$$

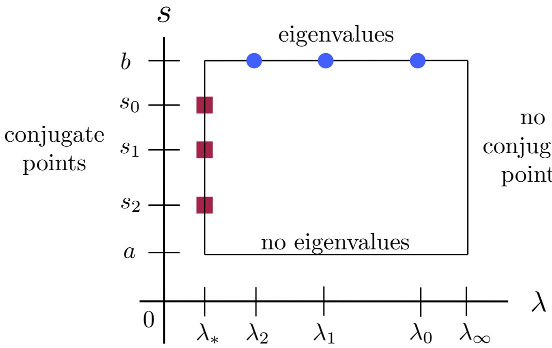
- Initial condition: $\theta(a;\lambda) = 0$; flow forward and see if $\theta(s;\lambda) \in \{j\pi\}$
- Fix $\lambda = \lambda_k$ to be an eigenvalue, so if $s = b$ we know $\theta(b;\lambda_k) = (k+1)\pi$
- Decrease s until you again land in $\{j\pi\}$, which is the conjugate point s_{k-1}.



- Process stops at largest s_0; θ no longer can complete one half-rotation

Case Study: Sturm-Liouville Theory

Relationship between eigenvalues and conjugate points:

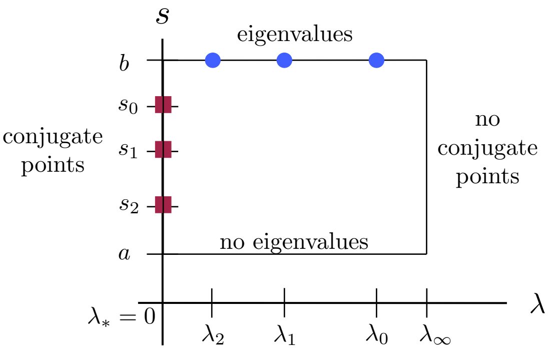


$\#\{$conjugate points for $\lambda = \lambda_*\} = \#\{$eigenvalues $\lambda > \lambda_*\}$

The angle θ is Arnol'd's angle ϕ that measures winding for the Maslov index!

Case Study: Sturm-Liouville Theory

To analyze stability, choose $\lambda_* = 0$:



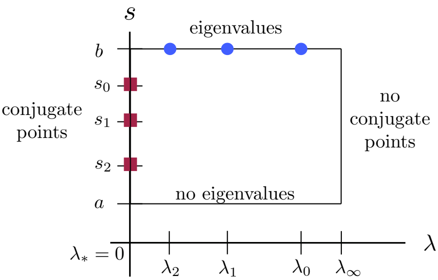
Number of conjugate points = number of unstable eigenvalues

It turns out this result can be viewed in terms of the Maslov index!

Create a map

$$\Phi : [0, \lambda_\infty] \times [a, b] \to \Lambda(2), \qquad (\lambda, s) \mapsto \begin{pmatrix} u(s; \lambda) \\ u_x(s; \lambda) \end{pmatrix}$$

Case Study: Sturm-Liouville Theory



$$\Phi : [0, \lambda_\infty] \times [a, b] \to \Lambda(2)$$

- The square is contractible in $\mathbb{R}^2$.
- Thus its image under Φ in $\Lambda(2)$ has Maslov index zero.
- There are no contributions to the index on the bottom or right.
- The contributions on the left and top are simple and of fixed respective sign, hence they are equal. Monotonicity is key here!

Summary so far

Stability is important in dynamical systems because it helps us understand which solutions we can expect to observe.

The Maslov index is a topological invariant that lets us count oscilllations in the Lagrangian Grassmanian.

One can use the Maslov index to count unstable eigenvalues of certain linear differential operators, and thus determine (in)stability.

Topological results in dynamics are powerful. For Sturm-Liouville theory, they provide information about stability without requiring detailed information about the stationary solution, or even about the underlying equation.

How generalizable is our case study?

- Can we generalize this to second-order systems? This would mean $(u, u_x) \in \mathbb{R}^{2n}$, so our phase space is no longer two-dimensional.
- How useful would this be? Is it easier to count conjugate points than to count unstable eigenvalues directly?
- Can we generalize this to higher-order equations? This is another way to obtain a non-planar phase space.
- Can we generalize this to multidimensional domains, eg $x \in \mathbb{R}^d$?

Generalizing Sturm-Liouville theory in various ways to study stability goes back to [Jones '88] and has also involved Bridges, Chardard, Chen, Cornwell, Cox, Deng, Dias, Fleurantin, Howard, Hu, Jaquette, Latushkin, Marangell, McQuighan, Pieper, Sukhtayev, ...

Second-order systems in one spatial dimension

$$v_t = v_{xx} + f(v), \qquad x \in \mathbb{R}, \qquad v \in \mathbb{R}^n$$

Key restrictive assumption:

$$f(v) = \nabla G(v), \qquad G : \mathbb{R}^n \to \mathbb{R}$$

Will imply linearized operator is self-adjoint and provide a symplectic structure.

Stationary solution $\varphi(x)$; suppose it is a pulse:

$$\lim_{x \to \pm\infty} \varphi(x) = \varphi_\infty$$

Eigenvalue equation:

$$\lambda u = u_{xx} + \nabla^2 G(\varphi(x))u = \mathcal{L}u, \qquad u \in L^2(\mathbb{R}, \mathbb{R}^n)$$

Natural assumption: the essential spectrum of $\mathcal{L}$ is stable. Equivalently,

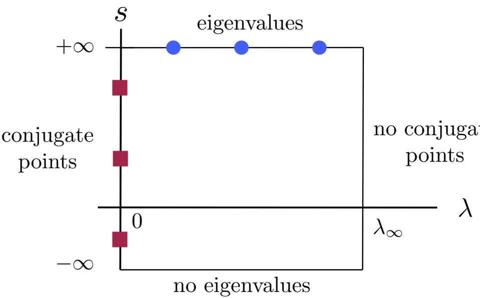
$$\nabla^2 G(\varphi_\infty) < 0.$$

Second-order systems in one spatial dimension

Main results of [B., Cox, Jones, Latushkin, McQuighan, Suhktayev '18]:

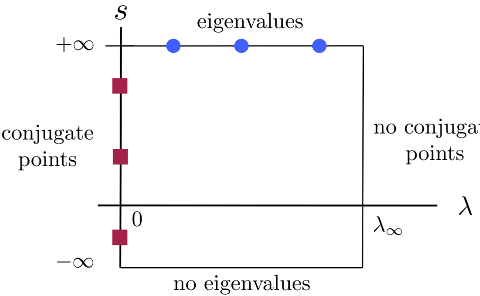
- Proved "square" relating eigenvalues to conjugate points.
- Proved a symmetric pulse is necessarily unstable, as in the scalar case.

$$\lambda u = u_{xx} + \nabla^2 G(\varphi(x))u = \mathcal{L}u, \qquad u \in \mathbb{R}^n$$



Second-order systems in one spatial dimension

$$\lambda u = u_{xx} + \nabla^2 G(\varphi(x))u = \mathcal{L}u, \qquad u \in \mathbb{R}^n, \qquad x \in \mathbb{R}$$



Main results of [B., Jaquette '22]:

- Framework for rigorously counting conjugate points via validated numerics.
- Key idea: prove conjugate points lie in $[-L, L]$, with explicit bounds on L.
- Applied to system of coupled bistable equations to demonstrate fronts can be either stable or unstable, unlike the scalar case.

Second-order systems in one spatial dimension

Eigenvalue equation:

$$\lambda u = u_{xx} + \nabla^2 G(\varphi(x))u = \mathcal{L}u, \qquad u \in \mathbb{R}^n, \qquad x \in \mathbb{R}$$

Write as a first-order system:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u \\ w \end{pmatrix} &= \begin{pmatrix} 0 & I \\ (\lambda - \nabla^2 G(\varphi(x))) & 0 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix} \qquad\qquad \begin{pmatrix} u \\ w \end{pmatrix} \in \mathbb{R}^{2n} \\
&= \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} (\lambda - \nabla^2 G(\varphi(x))) & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix} \\
&= J\mathcal{B}(x;\lambda)\begin{pmatrix} u \\ w \end{pmatrix}
\end{aligned}$$

Note that λ can be taken to be real and $\mathcal{B}(x;\lambda)$ is a symmetric matrix.

Can we develop a Sturm-Liouville-like theory for such eigenvalue problems?

- Arnol'd (1967, 1985) generalized the notion of phase to $\mathbb{R}^{2n}$ via the Maslov index and proved oscillation theorems.
- Can we connect his theory with eigenvalues?

Intersections of Langrangian planes in $\mathbb{R}^n$

Recall for each path of Lagrangian planes we get a path of frame matrices

$$\ell(t) = \left\{ \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} u : u \in \mathbb{R}^n \right\} \qquad \ell \Leftrightarrow \begin{pmatrix} A(t) \\ B(t) \end{pmatrix}$$

We then get a path of angles $\phi(t)$ and unitary matrixs $W(t)$

$$\mathrm{e}^{\mathrm{i}\phi(t)} = \det[\underbrace{(A(t) - \mathrm{i}B(t))(A(t) + \mathrm{i}B(t))^{-1}}_{=:W(t)}],$$

for which

$$\dim[\ker(W(t) + I)] = \dim(\ell(t) \cap \mathcal{D}),$$

where the reference plane is the Dirichlet plan

$$\mathcal{D} = \left\{ U = \begin{pmatrix} 0 \\ v \end{pmatrix} : v \in \mathbb{R}^n \right\} \in \Lambda(n)$$

This is very similar to looking for conjugate points!

Second-order systems in one spatial dimension

Back to our eigenvalue problem:

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u \\ v \end{pmatrix} = J\mathcal{B}(x;\lambda)\begin{pmatrix} u \\ v \end{pmatrix}$$

The assumption $\nabla^2 G(\varphi_\infty) < 0$ implies

$$J\mathcal{B}(\pm\infty;\lambda) = \begin{pmatrix} 0 & I \\ (\lambda - \nabla^2 G(\varphi_\infty)) & 0 \end{pmatrix}$$

is hyperbolic with stable/unstable eigenspace of dimension n. Therefore,

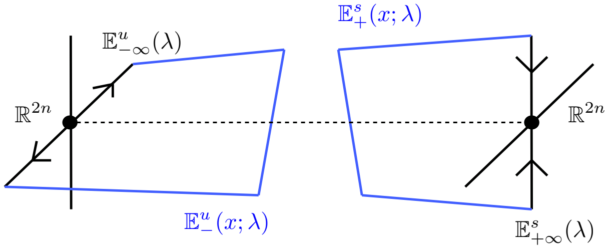
$$\dim(\mathbb{E}_\pm^{\mathrm{s,u}}(x;\lambda)) = n$$

In other words, the subspaces of solutions asymptotic to these stable/unstable eigenspaces at $\pm\infty$ must also have dimension n.

Second-order systems in one spatial dimension

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u \\ v \end{pmatrix} = J\mathcal{B}(x;\lambda)\begin{pmatrix} u \\ v \end{pmatrix}$$

For an eigenfunction, we need $(u,v)(x;\lambda) \in \mathbb{E}^{\mathrm{u}}_{-}(x;\lambda) \cap \mathbb{E}^{\mathrm{s}}_{+}(x;\lambda)$:



It turns out these are both paths of Lagrangian subspaces!

Alternatively, look for conjugate points:

$$\Phi(x;\lambda) = \mathbb{E}^{\mathrm{u}}_{-}(x;\lambda) \in \Lambda(n) \qquad \text{When is} \quad \mathbb{E}^{\mathrm{u}}_{-}(x;\lambda) \cap \mathcal{D} \neq \{0\}?$$

Summary

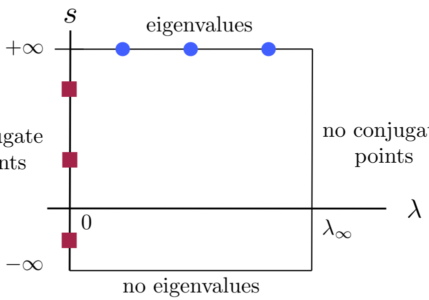
One can use the Maslov index to count unstable eigenvalues in scalar, second-order equations, and in systems of second-order equations with symplectic structure, and thus determine (in)stability.

Generalizations of these ideas also exist in higher-order equations, in higher spatial dimensions, and in the absence of monotonicity.

Other talks in this session:

- 2: Yuri Latushkin, "Fredholm determinants, Evans functions and Maslov indices for partial differential equations"
- 330: Emmanuel Fleurantin, "The Maslov Index and Noise-Induced Tipping"
- 4: Jonathan Jaquette, "Computer-Assisted-Proofs of Spectral Stability via Conjugate Points and the Maslov Index"
- 430: Graham Cox, "Hyperplane Maslov–Arnold spaces and the Turing instability"
- 5: Panel on Fundmaental Issues and Future Directions lead by Chris Jones.

Second-order systems in one spatial dimension



Some ideas in proof:

- Compactify domain $s \in (-\infty, \infty) \to \tilde{s} \in [-1, 1]$
- Path of Lagrangian subspaces $\ell(\tilde{s}; \lambda) = \mathbb{E}^{\mathrm{u}}_{-}(\tilde{s}; \lambda)$ around entire square
- Since $[-1, 1] \times [0, \lambda_\infty] \subset \mathbb{R}^2$ is a trivial loop, the Maslov index of $\ell(\tilde{s}; \lambda)$ around it must be zero.
- Show bottom and right side have no intersections.
- Prove monotonicity: determine sign of crossings on top $(-)$ and left $(+)$.
- Use "crossing form" to characterize Maslov index [Robbin, Salamon 1993]

Counting conjugate points via validated numerics

Key aspects of framework for counting conjugate points in [B., Jaquette '22]:

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u \\ v \end{pmatrix} = J\mathcal{B}(x;\lambda)\begin{pmatrix} u \\ v \end{pmatrix}, \qquad \mathbb{E}^{\mathrm{u}}_{-}(x_0;0)\cap\mathcal{D}\neq\{0\}?$$

- Use the facts that

$$E^{\mathrm{u}}_{-}(-\infty;0)\cap\mathcal{D}=\{0\}, \qquad |J\mathcal{B}(x;0)-J\mathcal{B}(-\infty;0)|\leq C\mathrm{e}^{-\eta|x|},$$

  to prove there is an L_- such that any conjugate point satisfies $x_0\geq -L_-$.
- Similarly prove there is an L_+ such that $x_0\leq L_+$.
- Obtain explicit bounds on $L_\pm$.
- Use the fact that

$$\mathbb{E}^{\mathrm{u}}_{-}(x_0;0)=\begin{pmatrix} A_1(x_0) \\ A_2(x_0) \end{pmatrix}\cap\mathcal{D} \qquad \text{iff} \qquad \det A_1(x_0)=0$$

  to count conjugate points on $[-L_-,L_+]$ by numerically finding zeros of the scalar-valued function $\det A_1(x)$.

Higher-order systems in one spatial dimension

Can we apply this method to study stability in Swift-Hohenberg?

$$u_t = -(1 + \partial_x^2)^2 u + f(u)$$

Together with H. Pieper, J. Jaquette, and J. Mireles-James we are working to

- Prove one can count unstable eigenvalues by counting conjugate points.
  Monotonicity is subtle; need to use a higher-order crossing form.
- Develop a framework to count conjugate points using validated numerics.
  Resonance in the vector bundle can't be dealt with using existing methods.

Some comments:

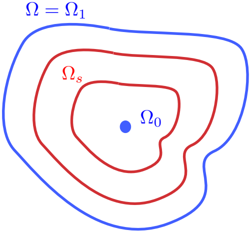
- Results of [Howard '21] on fourth-order systems don't apply here.
- Results in [Buffoni, Champneys, Toland '96] suggest one can determine spectral stability using local extrema, similar to the case of Sturm-Liouville theory. Would be very interesting to confirm this rigorously.

Systems in higher spatial dimensions

Eigenvalue problem:

$$\mathcal{L}u = \Delta u + V(x)u = \lambda u, \qquad x \in \Omega \subset \mathbb{R}^d$$
$$u|_{\partial\Omega} = 0$$

Family of domains [Smale 65]: $\{\Omega_s : 0 \leq s \leq 1\}$



Path of subspaces:

$$\ell(s;\lambda) = \left\{ \left(u, \frac{\partial u}{\partial n} \right) \Big|_{\partial\Omega_s} : u \in H^1(\Omega_s), \quad \Delta u + V(x)u = \lambda u, \quad x \in \Omega_s \right\}$$

Subspace of solutions on Ω_s with no reference to boundary conditions.

Systems in higher spatial dimensions

Dirichlet subspace:

$$\mathcal{D} = \left\{ \left(u, \frac{\partial u}{\partial n} \right) \Big|_{\partial\Omega} = \left(0, \frac{\partial u}{\partial n} \right) \Big|_{\partial\Omega} : u \in H^1(\Omega) \right\}$$

Reference plane determined by boundary conditions.

Conjugate point:

$$\ell(s;\lambda) \cap \mathcal{D} \neq \{0\}$$

Hilbert space

$$\mathcal{H} = H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega), \qquad \omega((f_1, g_1), (f_2, g_2)) = \langle g_2, f_1 \rangle - \langle g_1, f_2 \rangle$$

Both $\ell(s;\lambda)$ and $\mathcal{D}$ are in the Fredholm-Lagrangian Grassmannian of $\mathcal{H}$.

[Deng, Jones '11], [Cox, Jones, Latushkin, Suhktayev '16], ... relate conjugate points to eigenvalues via the Maslov index. See also related work using this family of domains to create a multi-dimensional spatial dynamics [B., Cox, Jones, Latushkin, Suhktayev '20, '21].

Summary, future work, and key open questions

Existing results:

- Sturm-Liouville Theory allows one to connect eigenvalues with conjugate points for scalar, second-order equations.
- The Maslov index allows for a generalization of this to second-order systems in one space dimension, and also to multiple spatial dimensions.
- Validated numerics can be used to efficiently and rigorously count conjugate points in second-order systems in one space dimension.

Current and future work on the Swift-Hohenberg equation:

- Prove one can count unstable eigenvalues by counting conjugate points.
- Develop a validated numerical framework for counting conjugate points.
- Use this to rigorously determine spectral stability using local extrema.

Open question: We are in need of applications! Either equations with symplectic structure where we can prove stability by counting conjugate points, or equations in higher spatial dimensions where a spatial dynamics framework might be useful for the analysis of existence, stability, and/or bifurcation.

Systems in one spatial dimension: $\mathbb{E}_-^u$ is Lagrangian

$$\frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = J\mathcal{B}(x;\lambda)\begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathcal{B}(x;\lambda)^* = \mathcal{B}(x;\lambda), \quad J^* = -J = J^{-1}$$

If $U, V \in \mathbb{E}_-^u(x;\lambda)$, then

$$\begin{aligned} \frac{d}{dx}\omega(U(x),V(x)) &= \langle U'(x), JV(x)\rangle + \langle U(x), JV'(x)\rangle \\ &= \langle J\mathcal{B}U(x), JV(x)\rangle + \langle U(x), J^2\mathcal{B}V(x)\rangle \\ &= \langle \mathcal{B}U(x), V(x)\rangle - \langle \mathcal{B}U(x), V(x)\rangle = 0. \end{aligned}$$

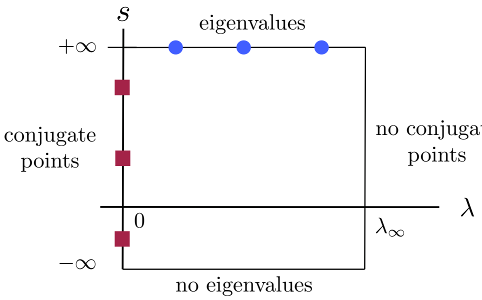
Moreover,

$$\lim_{x\to-\infty} U(x), V(x) = 0 \quad \Rightarrow \quad \lim_{x\to-\infty} \omega(U(x),V(x)) = 0$$

and so

$$\omega(U(x),V(x)) = 0 \qquad \forall x \in \mathbb{R}.$$

Systems in one spatial dimension: compactification



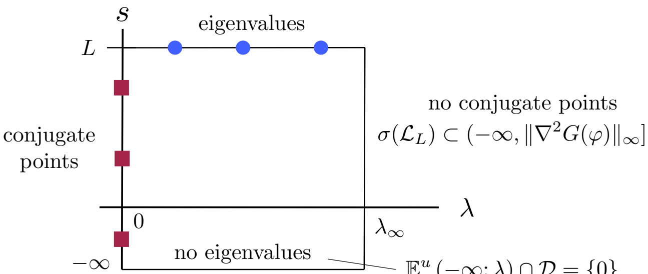
Compactify domain:

$$\sigma(s) = \tanh(s), \qquad s(\sigma) = \frac{1}{2}\ln\left(\frac{1+\sigma}{1-\sigma}\right), \qquad s \in [-\infty, \infty], \qquad \sigma \in [-1, 1]$$

Systems in one spatial dimension: additional step

First prove "square" on half-line with Dirichlet BCs:

$$\lambda u = u_{xx} + \nabla^2 G(\varphi(x))u = \mathcal{L}_L u, \qquad x \in (-\infty, L)$$
$$\mathrm{dom}(\mathcal{L}_L) = \{u \in H^2(-\infty, L) : u(L) = 0\}.$$



Extend result to full line $\mathbb{R}$: show for $L > L_\infty$ large,

- $\mathrm{Morse}(\mathcal{L}) = \mathrm{Morse}(\mathcal{L}_L)$
- Follows because you can approximate the point spectrum of an operator on $\mathbb{R}$ using a large subdomain.

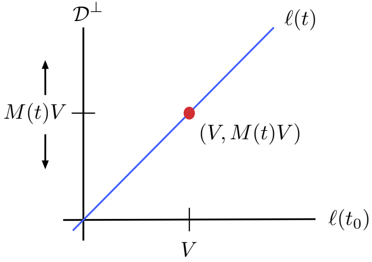
Systems in one spatial dimension: crossing form

$\ell : [a, b] \to \Lambda(n)$ path of Lagrangian planes, $\mathcal{D}$ reference plane. A crossing is a $t_0 \in [a, b]$ such that

$$\ell(t_0) \cap \mathcal{D} \neq \{0\}.$$

Generically $\ell(t)$ is transversal to $\mathcal{D}^\perp$ for all $t \in [t_0 - \epsilon, t_0 + \epsilon]$, and $\exists$ $M(t) : \ell(t_0) \to \mathcal{D}^\perp$ so that

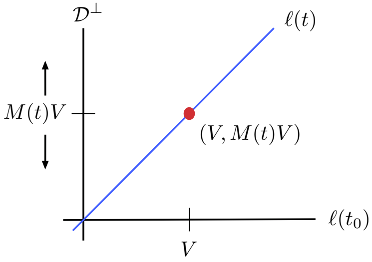
$$\ell(t) = \text{graph } M(t) = \{V + M(t)V : V \in \ell(t_0)\}$$



Crossing form [Robbin, Salamon '93]:

$$Q(U, V) = \frac{d}{dt}\omega(U, M(t)V)|_{t=t_0}, \qquad U, V \in \ell(t_0) \cap \mathcal{D}.$$

Systems in one spatial dimension: crossing form



Crossing form [Robbin, Salamon '93]:

$$Q(U,V) = \frac{d}{dt}\omega(U, M(t)V)|_{t=t_0}, \qquad U, V \in \ell(t_0) \cap \mathcal{D}.$$

- $Q \in \mathbb{R}^{k\times k}$ symmetric, where $k = \dim(\ell(t_0) \cap \mathcal{D})$.
- t_0 is regular if $\det Q \neq 0$; generic crossings are regular and isolated.
- Signature of Q:

$$\operatorname{sign} Q = n_+(Q) - n_-(Q),$$

$$n_\pm(Q) = \text{number of positive/negative eigenvalues}$$

Systems in one spatial dimension: crossing form

Maslov index for single crossing: if $t_0 \in [a_0, b_0]$ is the only crossing of ℓ with $\mathcal{D}$,

$$\mathrm{Mas}(\ell|_{[a_0,b_0]}, \mathcal{D}) = \begin{cases} -n_-(Q) & \text{if } t_0 = a_0 \\ \mathrm{sign}\, Q = n_+(Q) - n_-(Q) & \text{if } t_0 \in (a_0, b_0) \\ n_+(Q) & \text{if } t_0 = b_0 \end{cases}$$

- Endpoint convention is somewhat arbitrary; affects intermediate results but not our end result.
- Define Maslov index of a regular smooth path by defining it on segments around each crossing and summing.

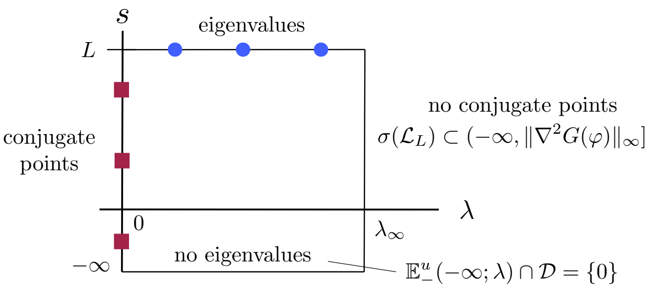
If all crossings of a path $\ell : [a, b] \to \Lambda(n)$ with $\mathcal{D}$ are positive, ie $Q > 0$, then

$$\mathrm{Mas}(\ell|_{[a,b]}, \mathcal{D}) = \sum_{a<t\le b} \dim(\ell(t) \cap \mathcal{D})$$

Similarly, if all crossings are negative, then

$$\mathrm{Mas}(\ell|_{[a,b]}, \mathcal{D}) = \sum_{a\le t<b} \dim(\ell(t) \cap \mathcal{D})$$

Systems in one spatial dimension: monotonicity



Key aspect of proof is showing monotonicity, ie crossings at conjugate points are positive, and crossings at eigenvalues are negative.

Path of Lagrangian planes: $\mathbb{E}^u_-(s; \lambda)$. Parameter is s or λ depending on side.

Negative crossings in λ: need to show for $s = L$ fixed

$$Q(U, V) = \frac{d}{d\lambda}\omega(U, M(\lambda)V)|_{\lambda=\lambda_0} < 0, \qquad U, V \in \mathbb{E}^u(L; \lambda_0) \cap \mathcal{D}.$$

Systems in one spatial dimension: monotonicity

Suffices to check that

$$Q(V,V) = \frac{d}{d\lambda}\omega(V, M(\lambda)V)|_{\lambda=\lambda_0} < 0, \qquad V \in \mathbb{E}^u(L;\lambda_0) \cap \mathcal{D}.$$

Let $W(L;\lambda) \in \mathbb{E}^u(L;\lambda)$ so that

$$W(L;\lambda_0) = V, \qquad W(L;\lambda) = V + M(\lambda)V.$$

We have

$$\begin{aligned} Q(V,V) &= \frac{d}{d\lambda}\omega(V, M(\lambda)V)|_{\lambda=\lambda_0} = \frac{d}{d\lambda}\omega(V, V + M(\lambda)V)|_{\lambda=\lambda_0} \\ &= \frac{d}{d\lambda}\omega(W(L;\lambda_0), W(L;\lambda))|_{\lambda=\lambda_0} = \omega(W(L;\lambda_0), W_\lambda(L;\lambda_0)). \end{aligned}$$

Recall:

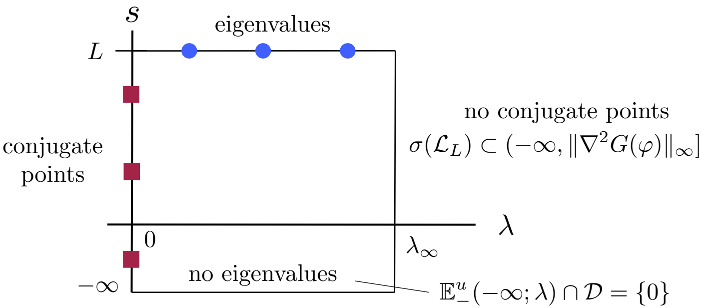
$$\frac{d}{dx}W = J\mathcal{B}(x;\lambda)W \qquad \Rightarrow \qquad \frac{d}{dx}W_\lambda = J\mathcal{B}(x;\lambda)W_\lambda + NW, \qquad N = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}.$$

Systems in one spatial dimension: monotonicity

$$Q = \omega(W(L;\lambda_0), W_\lambda(L;\lambda_0)), \qquad \frac{d}{dx} W_\lambda = J\mathcal{B}(x;\lambda)W_\lambda + NW.$$

$$\begin{aligned}
Q &= \langle -JW(L;\lambda_0), W_\lambda(L;\lambda_0)) \rangle = -\int_{-\infty}^{L} \frac{d}{dx} \langle JW(x;\lambda_0), W_\lambda(x;\lambda_0)) \rangle dx \\
&= -\int_{-\infty}^{L} [\langle J^2 \mathcal{B} W, W_\lambda)\rangle + \langle JW, J\mathcal{B}W_\lambda + NW\rangle] dx \\
&= -\int_{-\infty}^{L} \langle JW, NW \rangle dx \\
&= -\int_{-\infty}^{L} \left\langle \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} W, \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} W \right\rangle dx \\
&= -\int_{-\infty}^{L} (W_1(x;\lambda_0))^2 dx < 0.
\end{aligned}$$

Systems in one spatial dimension: monotonicity



This monotonicity implies:

$$
\begin{aligned}
0 &= \mathrm{Mas}(\mathbb{E}^u(x;\lambda)_{\mathrm{square}}, \mathcal{D}) \\
&= \mathrm{Mas}(\mathbb{E}^u(x;\lambda)_{\mathrm{left}}, \mathcal{D}) + \mathrm{Mas}(\mathbb{E}^u(x;\lambda)_{\mathrm{top}}, \mathcal{D}) + 0 + 0 \\
&= \{\text{number of conjugate points}\} - \{\text{number of eigenvalues}\}.
\end{aligned}
$$

Hence,

$$\{\text{number of conjugate points}\} = \{\text{number of eigenvalues}\} = \mathrm{Morse}(\mathcal{L}_L).$$

Systems in one spatial dimension: symmetric pulse instability

$$0 = \varphi_{xx} + \nabla G(\varphi(x)),$$

- Generically, $\varphi(x)$ will be unique as a solution (up to translation) asymptotic to the fixed point $\varphi_\infty = \lim_{x\to\pm\infty} \varphi(x)$
- Equation invariant under $x \to -x$, so $\varphi(-x)$ is also a solution. By uniqueness, we therefore have

$$\varphi(x) = \varphi(-x + \delta)$$

- This implies

$$\varphi(\delta/2 + x) = \varphi(\delta/2 - x) \qquad \forall \quad x \in \mathbb{R}.$$

- But then

$$\frac{d}{dx}\varphi(\delta/2 + x)|_{x=0} = \varphi(\delta/2 - x)|_{x=0} \qquad \Rightarrow \qquad \varphi_x(\delta/2) = 0.$$

- Since φ_x is a eigenfunction with eigenvalue $\lambda = 0$, we have

$$\begin{pmatrix} \varphi_x(x) \\ \varphi_{xx}(x) \end{pmatrix} \in \mathbb{E}^u_-(x;0) \qquad \Rightarrow \qquad \mathbb{E}^u_-(\delta/2, 0) \cap \mathcal{D} \neq \{0\}$$

which is our conjugate point.

Multiple space dimensions: Lagrangian subspace calculation

$$\ell(s;\lambda) = \left\{ \left(u, \frac{\partial u}{\partial n} \right) |_{\partial\Omega_s} : u \in H^1(\Omega_s), \quad \Delta u + V(x)u = \lambda u, \quad x \in \Omega_s \right\}$$

If $u, v \in \Phi$ then

$$\begin{aligned}
\omega(u,v) &= \left\langle \frac{\partial v}{\partial n}, u \right\rangle - \left\langle \frac{\partial u}{\partial n}, v \right\rangle \\
&= \int_{\partial\Omega} \left(\frac{\partial v}{\partial n} u - \frac{\partial u}{\partial n} v \right) dS \\
&= \int_{\Omega} \left((\nabla u \nabla v + u \Delta v) - (\nabla u \nabla v + v \Delta u) \right) dx \\
&= \int_{\Omega} \left(u(\lambda v - V v) - v(\lambda u - V u) \right) dx = 0.
\end{aligned}$$

Spatial dynamics in $\mathbb{R}^d$?

Does this suggest a "spatial dynamics" for $\mathbb{R}^d$? Consider

$$0 = \Delta u + F(x, u), \qquad x \in \Omega \subset \mathbb{R}^d$$

Family of domains parameterized by family of diffeomorphisms:

$$\psi_s : \Omega \to \Omega_s, \qquad s \in [0, 1], \qquad \Omega_1 = \Omega, \qquad \Omega_0 = \{x_0\}.$$

Define boundary data via

$$f(s; y) = u(\psi_s(y)), \qquad g(s; y) = \frac{\partial u}{\partial n}(\psi_s(y)), \qquad s \in [0, 1], \qquad y \in \partial\Omega$$

and trace map

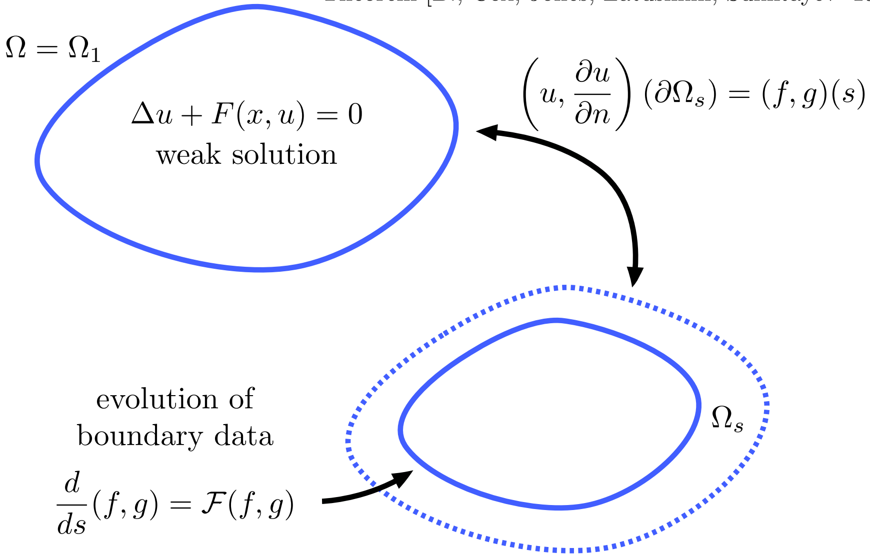
$$\mathrm{Tr}_s u = (f(s), g(s)).$$

Obtain an equivalent first-order system

$$\frac{d}{ds}\begin{pmatrix} f \\ g \end{pmatrix} = \mathcal{F}(f, g)$$

Spatial dynamics in $\mathbb{R}^d$?

Theorem [B., Cox, Jones, Latushkin, Sukhtayev '19]:



Spatial dynamics in $\mathbb{R}^d$?

So, rather than solving $\Delta u + F(x,u) = 0$, we can instead solve

$$\frac{d}{ds}\begin{pmatrix} f \\ g \end{pmatrix} = \mathcal{F}(f,g),$$

a somewhat technical and not-fun-to-look-at equation. What have we gained?

Consider $\Omega = \mathbb{R}^d$. We can choose to shrink the domain using spheres:

$$\Omega_s = \{x \in \mathbb{R}^d : |x| < s\}, \qquad s \in (0,\infty)$$

In terms of the polar coordinates $(r,\theta) \in (0,\infty) \times \mathbb{S}^{d-1}$,

$$\Delta u = u_{rr} + \frac{n-1}{r} u_r + \frac{1}{r^2} \Delta_{\mathbb{S}^{d-1}} u,$$

and our equation is now not so bad:

$$\frac{d}{ds}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -s^{-2}\Delta_{\mathbb{S}^{d-1}} & -(d-1)s^{-1} \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} + \begin{pmatrix} 0 \\ -F(t,\theta,s) \end{pmatrix}$$

Spatial dynamics in $\mathbb{R}^d$?

$$\frac{d}{ds}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -s^{-2}\Delta_{\mathbb{S}^{d-1}} & -(d-1)s^{-1} \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} + \begin{pmatrix} 0 \\ -F(s,\theta,f) \end{pmatrix}$$

- We have shown the linear part of this equation admits an exponential dichotomy (after rescaling time $s = e^{\tau}$).
- For $d = 3$, the dichotomy can be written explicitly in terms of the spherical harmonics of the Laplacian.
- This allows one to potentially construct solutions to the nonlinear equation that are not necessarily radially symmetric.
- We hope this will be a useful method for studying multidimensional waves and patterns.