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Appendix: Matrix Algebra

We give a brief summary of matrix algebra here. Matrices appear in examples in some chapters of the text and also are involved in several exercises.

A **matrix** is a rectangular array of numbers. For example, the array

$$\begin{bmatrix} 2 & -1 & 4 \\ 3 & 1 & 2 \end{bmatrix} \quad \textbf{(1)}$$

is a matrix having two rows and three columns. A matrix having m rows and n columns is an $m \times n$ matrix, so Matrix (1) is a 2×3 matrix. If $m = n$, the matrix is **square**. Entries in a matrix may be any type of number—integer, rational, real, or complex. We let $M_{m\times n}(\mathbb{R})$ be the set of all $m \times n$ matrices with real number entries. If $m = n$, the notation is abbreviated to $M_n(\mathbb{R})$. We can similarly consider $M_n(\mathbb{Z})$, $M_{2\times 3}(\mathbb{C})$, etc.

Two matrices having the same number m of rows and the same number n of columns can be added in the obvious way: we add entries in corresponding positions.

A1 Example In $M_{2\times 3}(\mathbb{Z})$, we have

$$\begin{bmatrix} 2 & -1 & 4 \\ 3 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -3 \\ 2 & -7 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 5 & -6 & 3 \end{bmatrix}. \quad \blacktriangle$$

We will use uppercase letters to denote matrices. If A, B, and C are $m \times n$ matrices, it is easily seen that $A + B = B + A$ and that $A + (B + C) = (A + B) + C$.

Matrix multiplication, AB, is defined only if the number of columns of A is equal to the number of rows of B. That is, if A is an $m \times n$ matrix, then B must be an $n \times s$ matrix for some integer s. We start by defining as follows the product AB where A is a

$1 \times n$ matrix and B is an $n \times 1$ matrix:

$$AB = [a_1 \quad a_2 \quad \cdots \quad a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n. \tag{2}$$

Note that the result is a number. (We shall not distinguish between a number and the 1×1 matrix having that number as its sole entry.) You may recognize this product as the *dot product* of vectors. Matrices having only one *row* or only one *column* are **row vectors** or **column vectors**, respectively.

A2 Example We find that

$$[3 \quad -7 \quad 2] \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} = (3)(1) + (-7)(4) + (2)(5) = -15.$$

▲

Let A be an $m \times n$ matrix and let B be an $n \times s$ matrix. Note that the number n of entries in each row of A is the same as the number n of entries in each column of B. The product $C = AB$ is an $m \times s$ matrix. The entry in the ith row and jth column of AB is the product of the ith row of A times the jth column of B as defined by Eq. (2) and illustrated in Example A2.

A3 Example Compute

$$AB = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 6 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 4 & 1 & -1 \\ -1 & 0 & 2 & 1 \end{bmatrix}.$$

Solution Note that A is 2×3 and B is 3×4. Thus AB will be 2×4. The entry in its second row and third column is

$$(\text{2nd row } A)(\text{3rd column } B) = [1 \quad 4 \quad 6] \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = 2 + 4 + 12 = 18.$$

Computing all eight entries of AB in this fashion, we obtain

$$AB = \begin{bmatrix} 2 & -2 & 9 & 6 \\ 1 & 17 & 18 & 3 \end{bmatrix}.$$

▲

A4 Example The product

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix}$$

is not defined, since the number of entries in a row of the first matrix is not equal to the number of entries in a column of the second matrix. ▲

For square matrices of the same size, both addition and multiplication are always defined. Exercise 10 asks us to illustrate the following fact.

Matrix multiplication is not commutative.

That is, AB need not equal BA even when both products are defined, as for $A, B \in M_2(\mathbb{Z})$. It can be shown that $A(BC) = (AB)C$ and $A(B + C) = AB + AC$ whenever all these expressions are defined.

We let I_n be the $n \times n$ matrix with entries 1 along the diagonal from the upper-left corner to the lower-right corner, and entries 0 elsewhere. For example,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is easy to see that if A is any $n \times s$ matrix and B is any $r \times n$ matrix, then $I_n A = A$ and $BI_n = B$. That is, the matrix I_n acts much as the number 1 does for multiplication when multiplication by I_n is defined.

Let A be an $n \times n$ matrix and consider a matrix equation of the form $AX = B$, where A and B are known but X is unknown. If we can find an $n \times n$ matrix A^{-1} such that $A^{-1}A = AA^{-1} = I_n$, then we can conclude that

$$A^{-1}(AX) = A^{-1}B, \qquad (A^{-1}A)X = A^{-1}B, \qquad I_n X = A^{-1}B, \qquad X = A^{-1}B,$$

and we have found the desired matrix X. Such a matrix A^{-1} acts like the reciprocal of a number: $A^{-1}A = I_n$ and $(1/r)r = 1$. This is the reason for the notation A^{-1}.

If A^{-1} exists, the square matrix A is **invertible** and A^{-1} is the **inverse** of A. If A^{-1} does not exist, then A is said to be **singular**. It can be shown that if there exists a matrix A^{-1} such that $A^{-1}A = I_n$, then $AA^{-1} = I_n$ also, and furthermore, there is only one matrix A^{-1} having this property.

A5 Example Let

$$A = \begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix}.$$

We can check that

$$\begin{bmatrix} -4 & 9 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -4 & 9 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus,

$$A^{-1} = \begin{bmatrix} -4 & 9 \\ 1 & -2 \end{bmatrix}.$$

▲

We leave the problems of determining the existence of A^{-1} and its computation to a course in linear algebra.

Associated with each square $n \times n$ matrix A is a number called the *determinant* of A and denoted by $\det(A)$. This number can be computed as sums and differences of certain products of the numbers that appear in the matrix A. For example, the determinant of the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$. Note that an $n \times 1$ matrix with real number entries can be viewed as giving coordinates of a point in n-dimensional Euclidean space $\mathbb{R}^n$. Multiplication of such a single column matrix on the left by a real $n \times n$ matrix A produces another such single column matrix corresponding to another point in $\mathbb{R}^n$. This multiplication on the left by A thus gives a map of $\mathbb{R}^n$ into itself. It can be shown that a piece of $\mathbb{R}^n$ of volume V is mapped by this multiplication by A into a piece of volume $|\det(A)| \cdot V$. This is one of the reasons that determinants are important.

The following properties of determinants for $n \times n$ matrices A and B are of interest in this text:

1. $\det(I_n) = 1$
2. $\det(AB) = \det(A)\det(B))$
3. $\det(A) \neq 0$ if and only if A is an invertible matrix
4. If B is obtained from A by interchanging two rows (or two columns) of A, then $\det(B) = -\det(A)$
5. If every entry of A is zero above the *main diagonal* from the upper left corner to the lower right corner, then det (A) is the product of the entries on this diagonal. The same is true if all entries below the main diagonal are zero.

EXERCISES A

In Exercises 1 through 9, compute the given arithmetic matrix expression, if it is defined.

1. $\begin{bmatrix} -2 & 4 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 1+i & -2 & 3-i \\ 4 & i & 2-i \end{bmatrix} + \begin{bmatrix} 3 & i-1 & -2+i \\ 3-i & 1+i & 0 \end{bmatrix}$

3. $\begin{bmatrix} i & -1 \\ 4 & 1 \\ 3 & -2i \end{bmatrix} - \begin{bmatrix} 3-i & 4i \\ 2 & 1+i \\ 3 & -i \end{bmatrix}$

4. $\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$

5. $\begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 5 & -3 \\ 2 & 1 & 6 \end{bmatrix}$

6. $\begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 7 \\ 3 & 1 \end{bmatrix}$

7. $\begin{bmatrix} i & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3i & 1 \\ 4 & -2i \end{bmatrix}$

8. $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}^4$

9. $\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}^4$

10. Give an example in $M_2(\mathbb{Z})$ showing that matrix multiplication is not commutative.

11. Find $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1}$, by experimentation if necessary.

12. Find $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{-1}$, by experimentation if necessary.

13. If $A = \begin{bmatrix} 3 & 0 & 0 \\ 10 & -2 & 0 \\ 4 & 17 & 8 \end{bmatrix}$, find det (A).

14. Prove that if $A, B \in M_n(\mathbb{C})$ are invertible, then AB and BA are invertible also.