MA 542: Modern Algebra II / Spring 2023
Homework assignment #1
Due ~~in class~~ Friday 1/27/23
(Either in class or by 10pm into envelope on my office door.)

Problem (10) added 23 January 23. Also some optional challenge problems.
Edit 1/25/23: Due time extended; minor sign typo changed in (11).

You must work in groups of two or three on this assignment.
Contact me by Monday 1/23/23 if you cannot find a group.

Review sections 1.1, 1.2, 1,3, 1.4. 2.1, 2.2 of BB (Beachy and Blair).
Solve the following exercises from BB.

(1) 1.1.5(a) and 1.1.7(a)

(2) 1.3.24

(3) 1.4.13

(4) 2.1.2

(5) 2.1.8(b,d,e) Explain!

(6) 2.1.15

(7) 2.1.20. (Showing that f is a function is about showing f is well-defined.)

(8) 2.2.2

(9) 2.2.3

Problem added 23 January.

(10) Let $R = \mathbb{Q}[\sqrt{2}]$ be the smallest subring of $\mathbb{C}$ containing $\mathbb{Q}$ and $\sqrt{2}$.

Formally, this means that R is the intersection of all the subrings of $\mathbb{C}$ that contain $\mathbb{Q}$ and $\sqrt{2}$. (Convince yourself that an intersection of subrings is still a subring; R is then the "least" subring of $\mathbb{C}$ containing $\mathbb{Q}$ and $\sqrt{2}$ in the sense that it's contained in *every* subring of $\mathbb{C}$ containing $\mathbb{Q}$ and $\sqrt{2}$.)

Also, let $G := \mathrm{Symm}(R) := \{f : R \to R \text{ bijective, } f \text{ preserves } + \text{ and } \cdot\}$.
Here by f preserving $+$ and $\cdot$ we of course mean that for every $x, y \in R$ we have $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$.

(a) Prove that G is a group under composition.

(b) Choose $f \in G$. Write out a beautiful complete proof of the fact that for every $a \in \mathbb{Q}$ we must have $f(a) = a$. (We sketched this argument in class.)

(c) Show that there is in fact an element f in G sending $\sqrt{2}$ to $-\sqrt{2}$. Explain.

Now, switch notation. As in class, let $\omega = e^{2\pi i/3}$; let $R := \mathbb{Q}[\sqrt[3]{2}, \omega]$ be the smallest subring of $\mathbb{C}$ containing $\mathbb{Q}$, $\sqrt[3]{2}$, and ω; and let $G := \mathrm{Symm}(R)$, all defined analogously.

(d) Show that there is an element $g \in G$ that sends ω to ω^2 and $\sqrt[3]{2}$ to itself.

One can also show that there is an element $h \in G$ that sends $\sqrt[3]{2}$ to $\omega\sqrt[3]{2}$ and ω to itself. (See optional challenge problem (11) below.)

(e) Determine the structure of G.

Optional challenge problems.
(Feel free to turn these in late — though at some point during the semester they may stop being challenging.)

(11) In problem (10), show that there is, in fact, an element $h \in G$ sending $\sqrt[3]{2}$ to $\omega\sqrt[3]{2}$ and ω to itself, as claimed.

(12) In our first class some of you argued that the ring
$$\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\} \subset \mathbb{C}$$
is a field, because every nonzero element $a + b\sqrt{2}$ has an inverse in $\mathbb{Q}[\sqrt{2}]$:
$$\frac{1}{a + b\sqrt{2}} = \frac{a}{a^2 - 2b^2} + \frac{-b}{a^2 - 2b^2}\sqrt{2}.$$
Is $\mathbb{Q}[\sqrt[3]{2}]$ also a field?

If yes, explain why the inverse of $1 + 5\sqrt[3]{2} + 7\sqrt[3]{4}$ is in $\mathbb{Q}[\sqrt[3]{2}]$.

If no, find a nonzero element of $\mathbb{Q}[\sqrt[3]{2}]$ that doesn't have an inverse in $\mathbb{Q}[\sqrt[3]{2}]$.

What about $R = \mathbb{Q}[\sqrt[3]{2}, \omega]$ from problem (10)?