MA 542: Modern Algebra II / Spring 2023
Homework assignment #3
Due Friday 2/24/23. Monday 2/27/23 is also ok.

Final version

Exercises from BB:

(1) 4.3.16

(2) 4.3.26

(3) 4.4.5

(4) 4.4.14

(5) 4.4.18

(6) 4.4.19

(7) 4.4.21

(8) 5.1.2

(9) 5.1.4

Additional exercises:

(10) Let F be any field containing $\mathbb{Z}_p$. Show that $\varphi(x) = x^p$ is a homomorphism from F to F. Show that φ is always injective. If F is finite, show that φ is surjective as well.

(11) Let R be the ring $\mathbb{Z}_5[x]/(x^2+1)$.

  (a) Show that the map $R \to \mathbb{Z}_5 \times \mathbb{Z}_5$ sending $[f(x)] \in R$ to $\big(f(2), f(3)\big)$ is well-defined. Show that it is an isomorphism of rings.

  (b) Find all the zero divisors of R. Explain.

(12) Show that there are infinitely many integer solutions (x, y) to the equation $x^2 - 2y^2 = 1$. (*Hint*: See (9).)

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(13) As in (10), let F be a field containing $\mathbb{Z}_p$. Let $f(x)$ be a polynomial in $\mathbb{Z}_p[x]$. If $\alpha \in F$ is a root of $f(x)$, show that α^p is also a root of $f(x)$.

(14) Let R be a finite ring. Show that every nonzero element of R is either a zero divisor or a unit (but never both!).

(15) BB 5.1.10

(16) BB 5.2.1

(17) BB 5.2.2

(18) BB 5.2.22

(19) **Optional challenge problem:** Let $\pi(x) \in \mathbb{Z}_p[x]$ be an irreducible polynomial of degree n. Show that $\pi(y)$ has a root in the field $F := \mathbb{Z}_p[x]/\big(\pi(x)\big)$. Use the ideas of (13) to show that $\pi(y)$ splits completely into distinct linear factors in $F[y]$.