MA 542: Modern Algebra II / Spring 2023
Homework assignment ~~#3~~ #4
Due Friday 3/17/23.

Final version

[Typo in (14a) corrected.]

(0) Read BB sections 5.2, 5.3, and 5.4.

(1) (a) (Review from MA 541.)
Show that a homomorphism $f : R \to S$ of abelian groups is injective if and only if $\ker f = \{0\}$. In particular, this is true if R and S are rings.

(b) If F is a field, S is a nonzero ring, and $f : F \to S$ is a ring homomorphism, show that f is automatically injective.
(In what form does this problem appear on HW #3?)

Sum, product, and intersection of ideals

(2) (a) 5.3.8
(b) For $n, m \in \mathbb{Z}$, compute $(n\mathbb{Z}) \cap (m\mathbb{Z})$.

(3) 5.3.16

(4) 5.3.19

More rings, ideals, homomorphisms, quotient rings from BB.

(5) 5.2.11

(6) 5.3.5

(7) 5.3.6

(8) 5.3.12

(9) 5.3.13

(10) 5.3.14

(11) 5.3.15

(12) 5.3.17

(13) (a) 5.3.22(a)
(b) 5.3.22(b)
(c) Is the ideal I of 5.3.22 principal? (Find a generator if so.)
Is it prime? Is it maximal?

More exercises

(14) Let $f : A \to B$ be a ring homomorphism. Fix an ideal $\mathfrak{a}$ of A and an ideal $\mathfrak{b}$ of B.
(a) Prove that $f^{-1}(\mathfrak{b})$ is an ideal of A ~~$\mathfrak{a}$~~.
(b) Let $A = \mathbb{Z}$ and $B = \mathbb{Z}_6$ and f is the reduction-mod-6 map.
List the ideals of B and their preimages in A.

(c) Prove that f induces an injective ring homomorphism $\tilde{f} : A/f^{-1}(\mathfrak{b}) \to B/\mathfrak{b}$.

(d) Is $f(\mathfrak{a})$ always an ideal of B? Either prove that it is or find a counterexample.

(15) Continuing the notation from (14)...

(a) If $\mathfrak{b}$ is a prime ideal of B, prove that $f^{-1}(\mathfrak{b})$ is a prime ideal of A.

(b) If $\mathfrak{b}$ is a maximal ideal of B, must $f^{-1}(\mathfrak{b})$ be a maximal ideal of A?
Either prove that this is always so or find a counterexample.

Added 5 March 2023

More from BB.

(16) 5.3.26

(17) 5.3.29

(18) 5.3.31
Note: $R \oplus S$ is BB's notation for the product $R \times S$ of rings.

(19) 5.3.32

(20) 5.4.4

These next few problems are optional. If you have time, do think about them — but there's no need to write anything up.

(21) 5.3.24

(22) (a) 5.3.27

(b) What is the characteristic of $\mathbb{Z}[i]/\langle 1+2i \rangle$?

(23) 5.3.28

(24) 5.4.6

(25) 5.4.10

(26) In class on 3/3 we proved Theorem 5.3.10: if $\mathfrak{p}$ is a nonzero prime ideal in a PID then $\mathfrak{p}$ is maximal. Where in the proof (same proof as in the book) did we use that $\mathfrak{p}$ was nonzero?

Optional challenge problems — these go a little further.

(27) Let $A = \mathbb{Z}/n\mathbb{Z}$ for some $n \geq 1$. Show that the following are equivalent for $a, b \in A$.

(a) a and b are associates in A.

(b) a and b are generators of the same ideal of A

(c) $\gcd(a, n) = \gcd(b, n)$

(d) There exists $u \in A^\times$ with $a = ub$.

(28) **Nonzero prime ideals of a PID are maximal: alternate argument pathway**
Let A be a commutative ring. A nonzero, nonunit $\pi \in A$ is called *prime* if, whenever $\pi \mid ab$ for $a, b \in A$, we have $\pi \mid a$ or $\pi \mid b$. Recall also that a nonzero, nonunit $\pi \in A$ is *irreducible* if any factorization of π involves units. (So the integers that we call *prime* in $\mathbb{Z}$ are *irreducible* by definition, and *prime* by the Fundamental Lemma (Lemma 1.2.5 in BB).)

(a) Show that a nonzero principal ideal $\langle a \rangle$ of A is prime if and only if a is a prime element.

(b) Show that a nonzero principal ideal $\langle a \rangle$ of A is maximal among principal ideals if and only if a is irreducible.

(An ideal $\langle a \rangle$ of A is maximal among principal ideals if $\langle a \rangle \neq A$ and no principal ideal sits properly between $\langle a \rangle$ and A: that is, if $\langle a \rangle \subseteq \langle b \rangle \subseteq A$ for some $b \in A$, then either $\langle a \rangle = \langle b \rangle$ or $\langle b \rangle = A$.)

(c) Show that every prime element is irreducible.

(d) Give an example to show irreducible elements need not be prime.

(*Suggestion:* Use the equation $2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$ in $\mathbb{Z}[\sqrt{-5}]$; the multiplicative norm $N(a + b\sqrt{-5}) = a^2 + 5b^2$ may be helpful.)

(e) If A is a PID, show that irreducible elements are always prime.

(Can you find more than one argument?)

(f) Give another proof that every nonzero prime ideal is maximal in a PID.