MA 542: Modern Algebra II / Spring 2023
Homework assignment #5
Due Friday 3/31/23.

Final version

Read BB sections 6.1 and A.7 and 6.2. Write up beautiful solutions to the following problems.

(1) BB 6.1.1(c)(d)(f)

(2) BB 6.1.2

(3) BB 6.1.5

(4) BB 6.1.10 (Note: in the setup, we have $K \subsetneq E \subseteq F = K(u)$)

(5) BB 6.1.12

(6) BB A.7.1

Also think about the following, but no need to turn in:

- BB 6.1.3
- BB 6.1.4
- BB 6.1.7
- BB 6.1.9

Added 3/25/23

(7) BB 6.2.1(b)(c)(d)(e)(f)

(8) BB 6.2.2

(9) BB 6.2.4

Think about the following, but no need to turn in:

- BB 6.2.5
- BB 6.2.9
- BB 6.2.10

Finally, following up on our in-class work...

(10) For each element α in some extension of the field K below, determine whether α is algebraic over K. If α is algebraic over K, find the minimal polynomial $f(x)$ of α, find a basis for $K(\alpha)$ over K, and factor $f(x)$ into a product of irreducibles in $K(\alpha)[x]$. If α is not algebraic over K, explain why not. You may assume that π is transcendental over $\mathbb{Q}$.

(a) $\alpha = \sqrt{\pi}$, $K = \mathbb{Q}(\pi)$
(b) $\alpha = \pi^2 + \pi^3$, $K = \mathbb{Q}$
(c) α is a primitive cube root of 1, $K = \mathbb{Z}_5$
(d) α is a primitive cube root of 1, $K = \mathbb{Z}_7$
(e) $\alpha = t^{1/3}$, $K = \mathbb{Q}(t)$
(f) $\alpha = t^{1/3}$, $K = \mathbb{Z}_7(t)$
(g) $\alpha = t^{1/3}$, $K = \mathbb{Z}_3(t)$

An element ζ is a *primitive* n^{th} *root of* 1 if $\zeta^n = 1$ but $\zeta^k \neq 1$ for every $1 \leq k < n$.
For example, in BB 6.2.1(f), the element ω is a primitive cube root of 1.

Optional challenge problems involving Zorn's lemma

Read about Zorn's lemma, for example, the introduction to Keith Conrad's first blurb on the topic. Then try the following problems.

(Don't get overwhelmed: (11) and (12) are plenty if this is your first encounter with Zorn's lemma!)

(11) **Any vector space has a basis:** Show this as follows.

Let V be a nonzero vector space over some field K.

(a) Consider the collections of linearly independent sets of vectors in V. Show that these form a nonempty poset under inclusion.

(b) Show that if $S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots$ is an increasing chain of linearly independent sets of vectors of V, then $S := \bigcup i \geq 1 S_i$ is a linearly independent set of vectors of V as well.

(c) Use Zorn's lemma to obtain a *maximal* linearly independent set M of vectors of V.

(d) Prove that M spans V. (If M spans a proper subspace of V, what can you do?)

(12) **Any ring has a maximal ideal:** Show this as follows.

Let A be a commutative ring. (In fact, A doesn't need to be commutative. But note that A does have to have a multiplicative identity: see (14) for counterexamples.)

(a) Show that the collection of proper ideals of A forms a nonempty poset under inclusion.

(b) Show that given an increasing chain $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathfrak{a}_3 \subseteq \cdots$ of proper ideals of A, the union $\mathfrak{a} := \bigcup_{i \geq i} \mathfrak{a}_i$ is also a proper ideal of A.

(It may be helpful here to characterize a proper ideal of A as any ideal of A not containing 1.)

(c) Use Zorn's lemma to conclude that A has maximal ideal.

(d) Now fix an auxiliary ideal $\mathfrak{b}$ of A. Prove that A has a maximal ideal containing $\mathfrak{b}$.

(13) **The intersection of prime ideals is the nilradical:** Recall that an element a of a ring A is *nilpotent* if $a^n = 0$ for some $n \geq 1$, and that the set N of nilpotent elements forms an ideal of A called the *nilradical*.

(a) Show (or recall) that a nilpotent element a of A is contained in every prime ideal of A. Conclude that N is contained in the intersection of the prime ideals of A.

To prove the converse — that the intersection of the prime ideals is contained in N — we (contrapositively) show that for any nonnilpotent element b of A, there is a prime ideal of A not containing b.

Fix a nonnilpotent b in A.

(a) Let Σ be the collection of ideals $\mathfrak{a}$ of A with the property that no positive power of b is in $\mathfrak{a}$. Show that Σ is a nonempty poset, ordered by inclusion. Show that the union of any chain of ideals in Σ is an ideal in Σ.

Zorn's lemma now implies that Σ has a maximal element $\mathfrak{m}$. Show that $\mathfrak{m}$ is a prime ideal as follows.

(b) Prove that for any $x \in A$ we have $x \notin \mathfrak{m}$ if and only if there exists a positive integer n with $b^n \in \mathfrak{m} + (x)$.

(c) Show that $x, y \notin \mathfrak{m}$ implies that $xy \notin \mathfrak{m}$. Conclude that $\mathfrak{m}$ is prime.

Finally, conclude that N is the intersection of all the prime ideals of A.

(14) **Rngs without** 1 **need not have maximal ideals:** Show through the following two examples that rngs (these are "rings" without the requirement that they have a multiplicative identity) need not have any maximal ideals.

(a) Define the rng $\tilde{\mathbb{Q}}$ as follows: as an additive group $\tilde{\mathbb{Q}} = \mathbb{Q}$, but we define multiplication by $a \cdot b = 0$ for any elements a, b. Show that $\tilde{\mathbb{Q}}$ is a rng. Show that an ideal of $\tilde{\mathbb{Q}}$ is the same as an additive subgroup of $\mathbb{Q}$. Show that $\tilde{\mathbb{Q}}$ has no maximal ideals.

(b) Consider the ring $A = \mathbb{Q}[x]$ localized at the ideal $\langle x \rangle$: that's the ring of rational functions $\frac{f(x)}{g(x)}$ where f, g are polynomials with $g(0) \neq 0$. Let $R := xA$ be the ideal of A generated by x. Prove that R is a rng under the addition and multiplication inherited from A. Prove that $R/xR \cong \tilde{\mathbb{Q}}$. Show that R has no maximal ideals.

(Note: in the notation of BB exercises 5.4.12 and 5.4.13, we have $R = M$ for $D = \mathbb{Q}[x]$ and $P = \langle x \rangle$; of course $A = D_P$.)