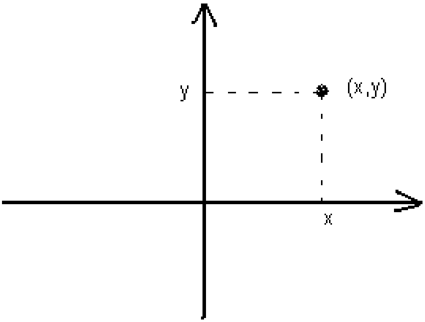
Vectors in 2D and 3D

Vectors

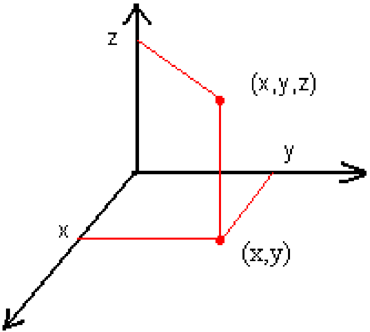
1. Three dimensional coordinates:

Point in plane: two perpendicular lines as reference:



Vectors in 2D and 3D

Point in space: three perpendicular lines as reference -



Vectors in 2D and 3D

$x - y$ plane plus z axis perpendicular to $x - y$ plane.

Coordinates of point $P:\ (x, y, z)$ indicated above

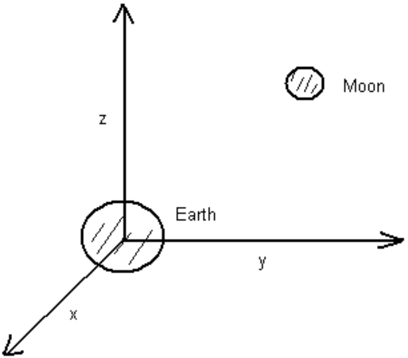
[e.g., three corner lines of the room]

[we get the three numbers (x, y, z) by dropping perpendiculars to the three axes]

Vectors in 2D and 3D

Ex 1: Calculation of an orbit to the moon:

[need to set up coordinate system and measure all points relative to this]



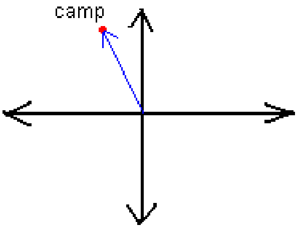
Vectors in 2D and 3D

[all mathematics is done by specifying position of the spacecraft and the moon relative to some coordinate system, say centered at the earth; that's how orbits are calculated at JPL and NASA]

Vectors in 2D and 3D

2. Vectors:

A vector is an arrow - it has direction and length. If you are hiking and say that you are 3 mi NNW of your camp you are specifying a vector.



Vectors in 2D and 3D

The precise mathematical statement is that:

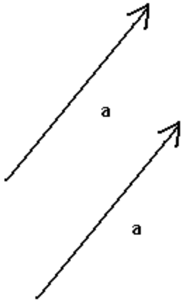
Geometric definition of vectors: A *vector* is a directed line segment. The length of a vector $\mathbf{v}$ is sometimes called its *magnitude* or the *norm* of $\mathbf{v}$. We will always abbreviate length by the symbol

$$\text{length of } \mathbf{v} = |\mathbf{v}|.$$

Vectors in 2D and 3D

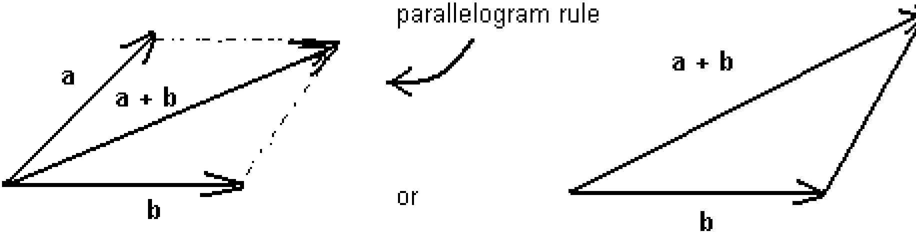
Two vectors are equal if they point in the same direction and have the same length:

[where the vector starts is not important]



Vectors in 2D and 3D

We can add vectors:



Vectors in 2D and 3D

And we can multiply vectors by real numbers (scalar multiplication):

If $\alpha > 0,$ then $\alpha\mathbf{a}$ is the vector in the direction of $\mathbf{a}$ whose length is $\alpha\mathbf{a}$.

If $\alpha < 0$ then $\alpha\mathbf{a}$ satisfies

$$\text{direction } (\alpha\mathbf{a}) = - \text{ direction } (\mathbf{a}).$$

$$|\alpha\mathbf{a}| = |\alpha||\mathbf{a}|\,.$$

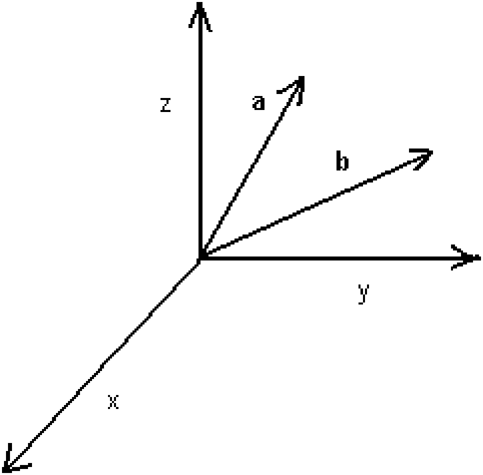
Vectors in 2D and 3D

Since we can add vectors and perform scalar multiplication, we can subtract two vectors:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + \textbf{-b}$$

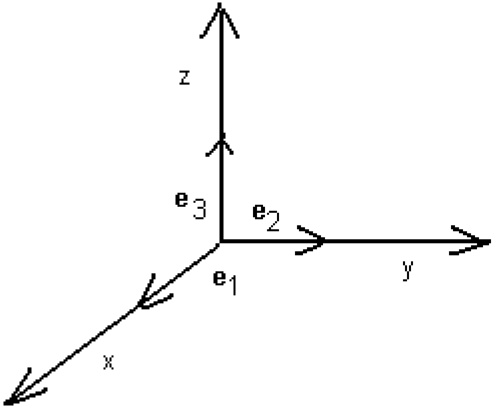
Now assume initial points of all vectors are located at the origin:

Vectors in 2D and 3D

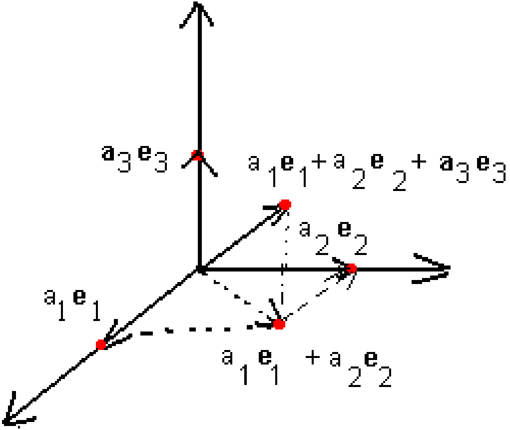


Vectors in 2D and 3D

Consider also the unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:



Vectors in 2D and 3D



Vectors in 2D and 3D

[We see $a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ and $a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$]

Notation: $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} =$

vector from origin to (a_1, a_2, a_3).

Given the vector $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$, we get by distance formula:

$$|\mathbf{a}| = \sqrt{(a_1 - 0)^2 + (a_2 - 0)^2 + (a_3 - 0)^2} = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Vectors in 2D and 3D

Analytic definition of vectors in 3 dimensions: A vector is an array $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ of numbers.

Vectors in 2D and 3D

3. Addition of vectors

Analytic addition of vectors:

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

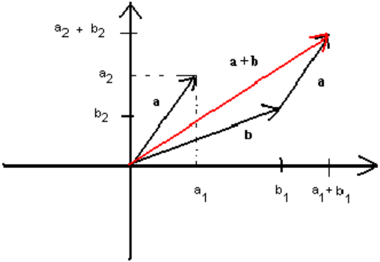


fig 4

Vectors in 2D and 3D

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$$

Simliarly in three dimensions:

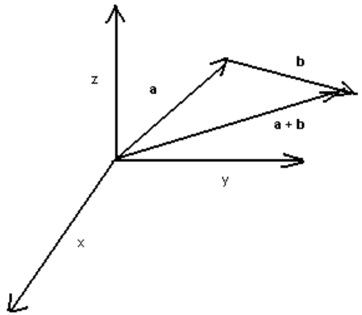


fig 5

Vectors in 2D and 3D

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\Rightarrow \qquad \mathbf{a} + \mathbf{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix} =$$

Moreover, can show similarly that:

$$\alpha \mathbf{a} = \begin{bmatrix} \alpha a_1 \\ \alpha a_2 \\ \alpha a_3 \end{bmatrix}$$

Example 2:

A light plane flies at a heading of due north (direction which airplane is pointed) at air speed (speed relative to the air) of 120 km/hr in a wind blowing due east at 50 km/hr. What direction and speed does the plane move at relative to the ground?

A: Define the velocity of the airplane as the vector **v** whose length is the speed of the plane and whose direction is the direction of the plane:

Vectors in 2D and 3D

Note: wind velocity is $\begin{bmatrix} 50 \\ 0 \end{bmatrix}$; plane's air velocity (relative to wind) is $\begin{bmatrix} 0 \\ 120 \end{bmatrix}$

Also: every hour plane flies 120 kilometers north and 50 kilometers east.

Thus direction of the plane is same as $\begin{bmatrix} 50 \\ 120 \end{bmatrix}$

Vectors in 2D and 3D

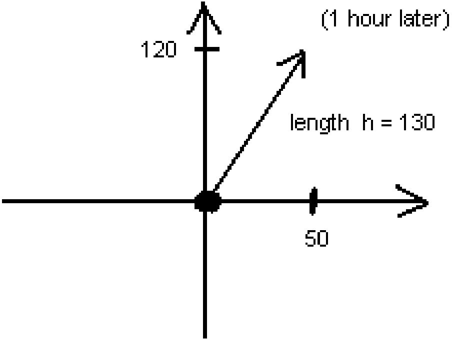


fig 7

What is the speed: $\sqrt{120^2 + 50^2} = 130.$

Vectors in 2D and 3D

Thus speed of airplane is 130 = length of $\begin{bmatrix} 50 \\ 120 \end{bmatrix}$;

direction of airplane is same as $\begin{bmatrix} 50 \\ 120 \end{bmatrix}$

$\Rightarrow$ velocity of airplane is vector

$$\begin{bmatrix} 50 \\ 120 \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 120 \end{bmatrix}$$ = wind velocity + air velocity of plane

Vectors in 2D and 3D

Moral: when we take an object moving at velocity $\mathbf{v}_1$ (air velocity) in a medium which is moving at velocity $\mathbf{v}_2$ (wind velocity), the total velocity of the object is $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$.

4. Properties of vectors:

Theorem: *If* $\mathbf{a}$,$\mathbf{b}$, *and* $\mathbf{c}$ *are vectors (in two or three dimensions) and* d *and* e *are scalars, then:*

(a) $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$ *(commutativity)*

(b) $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$ *(associativity)*

(c) $d(\mathbf{a}+\mathbf{b})=d\mathbf{a}+d\mathbf{b}$ *(distributivity)*

(d) $(d+e)\mathbf{a}=d\mathbf{a}+e\mathbf{a}$ *(distributivity)*

(e) *There is a unique vector* $\mathbf{0}$ *such that* $\mathbf{a}+\mathbf{0}=\mathbf{a}$ *for all vectors* $\mathbf{a}$

(f) *For every vector* $\mathbf{a}$ *there is a unique vector* $-\mathbf{a}$ *such that* $\mathbf{a}+-\mathbf{a}=0$

(g) $d(e\mathbf{a})=(de)\mathbf{a}$

(h) $1\cdot\mathbf{a}=\mathbf{a}$

Properties of Vectors

[other properties you would expect are listed in the book; they follow from the above properties]

Proof: (a): Assume that

$$\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Note that the entries in **a** have the general form a_i for $i = 1$ through n.

Properties of Vectors

We will use shorthand by writing **a** in terms of the general term in **a**:

$$\mathbf{a} = (a_i).$$

Similarly,

$$\mathbf{b} = (b_i).$$

Thus

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} = \begin{bmatrix} b_1 + a_1 \\ \vdots \\ b_n + a_n \end{bmatrix} = \mathbf{b} + \mathbf{a}.$$

Properties of Vectors

In shorthand the above could be written:

$$\mathbf{a} + \mathbf{b} = (a_i) + (b_i) = (a_i + b_i) = (b_i + a_i) = (b_i) + (a_i) = \mathbf{b} + \mathbf{a}.$$

(b)

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = (\,(a_i) + (b_i)) + (c_i) = (a_i + b_i) + (c_i)$$

$$= ((a_i + b_i) + c_i) = (a_i + (b_i + c_i)) = (a_i) + (b_i + c_i)$$

$$= a_i + ((b_i) + (c_i)) = \mathbf{a} + (\mathbf{b} + \mathbf{c}) \text{ as desired.}$$

Properties of Vectors

As an exercise, write out the proof in full, i.e. replace

$$(a_i) \rightarrow \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$(a_i + b_i) \rightarrow \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix},$$

etc., to see how it looks.

Properties of Vectors

[Note it is remarkable that if you replace the word vector with the word matrix, the same statements as above are all still true.]

Vectors in n dimensions

Analytic definition of vectors in n dimensions: A vector is a vertical array of n numbers:

$$\mathbf{v} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

Note: all definitions of analytic operations on vectors in 3 dimensions hold for vectors in n dimensions.

Vectors in n dimensions

Can easily see that all properties (1)-(8) of vectors in 3 and 2 dimensions carry over to vectors in n dimensions; proofs are identical.

[geometric visualization of vectors in n dimensions is not necessary at this point]

Vector spans

5. Spans of vectors

Def 6: We define $\mathbb{R}^3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \; x_i \in \mathbb{R} \right\}$

where $\mathbb{R}$ means the set of all real numbers. Thus $\mathbb{R}^3$ is all 3-tuples of real numbers.

Def 7: A *linear combination* of two vectors **a** and **b** is a sum

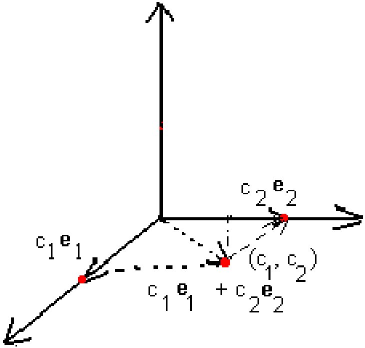
$$c_1\mathbf{a} + c_2\mathbf{b}$$

for constants c_1 and c_2.

Linear combination for larger collection of vectors works the same way.

Vector spans

Ex: Consider two vectors $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{e_1}; \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{e}_2$:



Vector spans

Then:

$$c_1\mathbf{a} + c_2\mathbf{b} = c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}c_1\\c_2\\0\end{bmatrix}.$$

Note as c_1 and c_2 vary, this covers *all* points in the $x - y$ plane.

We say *span* of **a** and **b** $=$ all possible linear combinations of **a** and **b** $=$ all vectors in $x - y$ plane.

Vector spans

Generally can show: span of two vectors $=$ all vectors contained in the plane of the first two.

General definition: The *span* of a collection of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is the collection of all possible linear combinations

$$\text{span}\{\mathbf{v_1}, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n : \ c_i \in \mathbb{R}\}.$$

Above $\in$ means "is an element of".

6. Matrices and linear systems of equations:

Return to previous problem: Coal, Steel, Electricity:

[NOTE: There is an example of Leontief's economic model in the book; the equations there are DIFFERENT than here; they guarantee that the amount used for the production of each commodity (coal, steel, electricity) ends up equalling the value of the commodity produced; in our variation of Leontief's equations we have (non-zero) targets for net production of each commodity, and find production levels to attain those targets]

Matrices and linear systems

To make \$$1$ of coal, takes no coal, \$.10 of steel, \$.10 of electricity.

To make \$$1$ of steel, it takes \$.20 of coal, \$.10 of steel, and \$.20 of electricity.

To make \$$1$ of electricity, it takes \$.40 of coal; \$.20 of steel, and \$.10 of electricity.

Let

$$\begin{aligned} x_1 &= \text{ total amount of coal produced} \\ x_2 &= \text{ total amount of steel produced} \\ x_3 &= \text{ total amount of electricity produced} \end{aligned}$$

Matrices and linear systems

Goals of economy:

If we want the economy to output $1 billion of coal, $.7 billion of steel, and $2.9 billion of electricity, how much coal, steel, and electricity will we need to use up? I.E., what will x_1, x_2, x_3 be?

Recall the equations:

$$x_1 - .2x_2 - .4x_3 = 1$$

(*) $$-.1x_1 + .9x_2 - .2x_3 = .7$$

$$-.1x_1 - .2x_2 + .9x_3 = 2.9$$

Matrices and linear systems

give us x_i for $i = 1, 2,$ and 3.

Note: system (*) equivalent to a single matrix equation:

$$\begin{bmatrix} 1 & -.2 & -.4 \\ -.1 & .9 & -.2 \\ -.1 & -.2 & .9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ .7 \\ 2.9 \end{bmatrix}$$

Matrices and linear systems

Check:

$\rightarrow$

$$\begin{bmatrix} x_1 & -\ .2x_2 & -\ .4x_3 \\ -\ .1x_1 & +\ .9x_2 & -\ .2x_3 \\ -\ .1x_1 & -\ .2x_2 & +\ .9x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ .7 \\ 2.9 \end{bmatrix}$$

$\Rightarrow$

$$x_1 - .2x_2 - .4x_3 = 1$$

(*)

$$-\ .1x_1 + .9x_2 - .2x_3 = .7\,,$$

$$-\ .1x_1 - .2x_2 + .9x_3 = 2.9$$

Matrices and linear systems

= original system of equations.

[Very compact form for writing equations.]

If we call

$$A = \begin{bmatrix} 1 & -.2 & -.4 \\ -.1 & .9 & -.2 \\ -.1 & -.2 & .9 \end{bmatrix},$$

Matrices and linear systems

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ .7 \\ 2.9 \end{bmatrix},$$

then the equation reads:

$$A\mathbf{x} = \mathbf{b}.$$

Thus can concisely describe systems using matrices.

Matrices and linear systems

Generally: if we have

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + \ldots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

Matrices and linear systems

Can be described by $A\mathbf{x} = \mathbf{b}$ where:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = [\, a_{ij} \,]$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Matrices and linear systems

[now we can concisely write any system of equations].

7. Matrix multiplication and vectors.

Ex 1: System from above:

$$\begin{bmatrix} 1 & -.2 & -.4 \\ -.1 & .9 & -.2 \\ -.1 & -.2 & .9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ .7 \\ 2.9 \end{bmatrix}$$

$$A\mathbf{x} = \mathbf{b}.$$

General form:

Matrix multiplication and vectors

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Write:

$$A = [\mathbf{a}_1 \; \mathbf{a}_2 \; \mathbf{a}_3]$$

where

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -.1 \\ -.1 \end{bmatrix}; \qquad \mathbf{a}_2 = \begin{bmatrix} -.2 \\ .9 \\ -.2 \end{bmatrix}; \quad \mathbf{a}_3 = \begin{bmatrix} -.4 \\ -.2 \\ .9 \end{bmatrix}.$$

Matrix multiplication and vectors

Focus on left side:

$$A\mathbf{x} = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Matrix multiplication and vectors

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ a_{31}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \\ a_{32}x_2 \end{bmatrix} + \begin{bmatrix} a_{13}x_3 \\ a_{23}x_3 \\ a_{33}x_3 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

Matrix multiplication and vectors

$$= x_1\mathbf{a_1} + x_2\mathbf{a}_2 + x_3\mathbf{a}_3$$

Conclusion:

$$A\mathbf{x} = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\mathbf{a_1} + x_2\mathbf{a}_2 + x_3\mathbf{a}_3$$

Ex 2:

$$A = \begin{bmatrix} 1 & -.2 & -.4 \\ -.1 & .9 & -.2 \\ -.1 & -.2 & .9 \end{bmatrix}.$$

Matrix multiplication and vectors

Then

$$A\mathbf{x} = \begin{bmatrix} 1 & -.2 & -.4 \\ -.1 & .9 & -.2 \\ -.1 & -.2 & .9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ -.1 \\ -.1 \end{bmatrix} + x_2 \begin{bmatrix} -.2 \\ .9 \\ -.2 \end{bmatrix} + x_3 \begin{bmatrix} -.4 \\ -.2 \\ .9 \end{bmatrix}.$$

[another way of writing it]

Matrix multiplication and vectors

Note that $A\mathbf{x} = \mathbf{b}$ is:

$$x_1 \begin{bmatrix} 1 \\ -.1 \\ .1 \end{bmatrix} + x_2 \begin{bmatrix} -.2 \\ .9 \\ -.2 \end{bmatrix} + x_3 \begin{bmatrix} -.4 \\ -.2 \\ .9 \end{bmatrix} = \begin{bmatrix} 1 \\ .7 \\ 2.9 \end{bmatrix}.$$

8. Spanning and equations:

Ex 3: Consider question: What is the span of the vectors

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \mathbf{a}_3 = \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}?$$

Answer: Find all vectors $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ obtained as:

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 = \mathbf{b} \qquad (1)$$

for some c_1, c_2, c_3.

Spanning and equations

Equations read:

$$[\mathbf{a}_1 \mathbf{a_2}\, \mathbf{a}_3] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

or

$$A\mathbf{c} = \mathbf{b} \quad \text{where} \quad A = [\mathbf{a}_1\, \mathbf{a}_2\, \mathbf{a}_3], \;\; \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

or

Spanning and equations

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Question: for what b_1, b_2, b_3 is there a solution c_1, c_2, c_3 above.

[System of 3 eqn's in 3 unknowns.]

Spanning and equations

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ -1 & 0 & -2 & b_2 \\ 0 & 1 & -1 & b_3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 0 & 2 & -2 & b_1 + b_2 \\ 0 & 1 & -1 & b_3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 0 & 1 & -1 & b_1/2 + b_2/2 \\ 0 & 1 & -1 & b_3 \end{array}\right]$$

Spanning and equations

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & | & b_1 \\ 0 & 1 & -1 & | & b_1/2 + b_2/2 \\ 0 & 0 & 0 & | & b_3 - b_1/2 - b_2/2 \end{bmatrix} \qquad (2)$$

Note: exists solution iff $b_3 - b_1/2 - b_2/2 = 0$.
Conclude solution does NOT exist for all **b**.

Note that:
equation (1) has solution for ALL **b**

$\Leftrightarrow$ every vector $\mathbf{b} \in \mathbb{R}^3$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$

$\Leftrightarrow$ $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ span $\mathbb{R}^3$.

Spanning and equations

[Note also: if left side of (2) had a pivot entry on last row, we wouldn't have problem with no solutions].

Conclude: also:

equation (1) has a solution for ALL **b** $\Leftrightarrow$ echelon form of A has a pivot entry in every row.

Theorem 4: *The following are equivalent:*
(a) The columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ *of* A *span* $\mathbb{R}^3$
(b) $A\mathbf{x} = \mathbf{b}$ *has a solution for each* $\mathbf{b} \in \mathbb{R}^3$
(c) A *has a pivot position in every row.*

Spanning and equations

Proof: In book.

More generally, this theorem holds for any set of vectors:

Theorem 5: *The following are equivalent for a* $m \times n$ *matrix* A*:*

(a) The columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ *of a matrix* A *span* $\mathbb{R}^m$

(b) $A\mathbf{x} = \mathbf{b}$ *has a solution for each* $\mathbf{b} \in \mathbb{R}^n$

(c) A *has a pivot position in every row.*

Spanning and equations

Note this gives an algorithm for checking whether vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ span $\mathbb{R}^n$ - form matrix A and check if every row has a pivot position.

[we now start material from section 1.5]

Properties of matrix-vector products:

Theorem 6: *If A is a matrix and* **u** *and* **v** *are vectors and c is a scalar, then:*

(a) $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$

(b) $A(c\mathbf{u}) = cA(\mathbf{u})$.

Proof: (a) :

$$A(\mathbf{u}+\mathbf{v}) = [\mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_n] \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

$$= (u_1 + v_1)\mathbf{a}_1 + (u_2 + v_2)\mathbf{a}_2 + \ldots + (u_n + v_n)\mathbf{a}_n$$

$$= (u_1\mathbf{a}_1 + \ldots + u_n\mathbf{a}_n) + (v_1\mathbf{a}_1 + \ldots + v_n\mathbf{a}_n)$$

$$= A\mathbf{u} + A\mathbf{v}.$$

(b) In book - proof is similar.