MA573 - Fall 2019 Homework 6 - Due October 11th

Hirsh-Smale-Devaney Problems

Chapter 5: 8

Chapter 6: 1(a,c,f), 2, 3, 7

Note: for part 7(d), you do not need to rigorously prove anything. You should explain, to the best of your ability, your conclusions about periodicity of solutions)

Non-textbook Problem(s):

Problem 1: *(Projective coordinates) Consider the linear system*

$$\begin{aligned} x' &= x, \\ y' &= -y. \end{aligned} \tag{0.1}$$

For any line $\ell \subset \mathbb{R}^2$ through the origin, there exists a vector $V = (v_1, v_2)^T \in \mathbb{R}^2$ such that

$$\ell = \mathrm{Span}\{V\} = \{c(v_1, v_2)^T \,|\, c \in \mathbb{R}\}.$$

(i) Let ϕ_t be the flow of the linear system (0.1) above. As $t \to \pm\infty$ what does the set $\phi_t(\ell)$ look like for a given t and approach as $t \to +\infty$? In other words, what does the set $\{\phi_t(x_0, y_0) \| \, (x_0, y_0) \in \ell\}$ look like for a given t? What does it look like as $t \to +\infty$? If the solution blows up, what is it's asymptotic slope? Does it matter what line $\ell \subset \mathbb{R}^2$ one takes?

(ii) Now we derive a 1-D equation to characterize the behavior found above. Set $z = y/x$, derive a differential equation for z. That is find an $f(z)$ such that $\frac{d}{dt} z = f(z)$. Draw the phase line of this equation and relate the dynamics (namely the equilibria, and asymptotics of solutions) observed here to those of the line dynamics in part (i) above.

(iii) (optional one to think about): Now consider the same problem but in three dimensions: That is let

$$\begin{aligned} x' &= x, \\ y' &= 2y, \\ z' &= -z. \end{aligned}$$

and consider how lines $\ell \subset \mathbb{R}^3$ evolve under the flow.