MA776 - Spring 2022 Homework 4 - Due Friday, April 15th

Please scan or type your homework and email it to me by the due date. Homeworks will be graded electronically and emailed back to you.

Evans Problems

Chapter 5: Problems 14,17,20,21

Remark: Problem 21 shows which spaces $H^s(\mathbb{R}^n)$ are Banach algebras, a useful characteristic when studying nonlinear PDE with polynomial nonlinearities.

Additional Problems :

Problem 1: Let $\Omega = (-1, 1) \subset \mathbb{R}$, and $u(x) = |x|$. Show that $u \in W^{1,p}(\Omega)$ but $u \notin W^{2,p}(\Omega)$ for $p \in (1, \infty)$.

Problem 2: (a): (Ehrling's Lemma): Let X, Y, Z be Banach spaces such that X is compactly embedded in Y and Y is continuously embedded in Z. Prove that for every $\epsilon > 0$ there exists a constant $C(\epsilon) > 0$ such that

$$\|u\|_Y \leq \epsilon \|u\|_X + C(\epsilon)\|u\|_Z,$$

for all $u \in X$. (Hint: go by contradiction with a sequence $\{u_n\} \in X$ with $\|u_n\|_X = 1$)

(b): Let Ω be bounded with C^1 boundary. Apply part (a) to the spaces $X = H^k(\Omega), Y = H^{k-1}(\Omega), Z = L^1(\Omega)$ and show that the standard norm $\|u\|_{H^k}^2 = \sum_{|\alpha|\leq k} \|D^\alpha u\|_{L^2}^2$ is equivalent to the following norm

$$\|u\|_{H^k,*}^2 := \sum_{|\alpha|=k} \|D^\alpha u\|_{L^2}^2 + \|u\|_{L^1}^2.$$

Problem 3: Let $u, v \in H^1(\mathbb{R})$. Prove that

$$\int_{-\infty}^{\infty} u(x)v'(x)dx = -\int_{-\infty}^{\infty} u'(x)v(x)dx.$$

Problem 4: Let Ω be open, bounded with C^1 boundary. Show that every weakly convergent sequence in $H^1(\Omega)$ converges strongly in $L^2(\Omega)$. (See the appendices for more on weak convergence).