DIHEDRAL ARTIN REPRESENTATIONS AND CM FIELDS

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Abstract. For a fixed CM field K with maximal totally real subfield F, we consider isomorphism classes of dihedral Artin representations of F which are induced from K, distinguishing between those which are "canonically" induced from K and those which are "noncanonically" induced from K. The latter can arise only for Artin representations with image isomorphic to the dihedral group of order 8. We show that asymptotically, the number of noncanonically induced isomorphism classes is always comparable to and in some cases exceeds the number of canonically induced ones.

A standard problem in arithmetic statistics is to count the Galois extensions of F of some fixed type, where F is a given number field. Less standard perhaps are comparisons between the asymptotic formulas obtained for two different counting problems associated to F, but such comparisons can be instructive. A case in point is the recent paper of Friedrichsen and Keliher [7]. Let D_m denote the dihedral group of order $2m$ and S_m the symmetric group on m letters. Friedrichsen and Keliher compare the asymptotic formula for D_4-extensions due to Cohen, Diaz y Diaz, and Olivier [5] with the asymptotic formula for S_4-extensions due to Bhargava, Shankar, and Wang [4]. In both cases, the objects counted are nonnormal quartic extensions M of F with $\mathbf{N}\mathfrak{d}_{M/F} \leqslant x$ and $\mathrm{Gal}(\widetilde{M}/F) \cong D_4$ or $\mathrm{Gal}(\widetilde{M}/F) \cong S_4$ respectively, where $\widetilde{M}$ denotes the normal closure of M over F, $\mathbf{N}$ is the absolute norm, and $\mathfrak{d}_{M/F}$ is the relative different ideal of M/F. (Note that $\mathbf{N}\mathfrak{d}_{M/F}$ coincides with the absolute norm of the relative discriminant ideal of M/F.) In contrast to the case $F = \mathbb{Q}$, where Bhargava's fundamental work [3] shows that $\mathrm{Gal}(\widetilde{M}/\mathbb{Q}) \cong S_4$ for approximately 83% of quartic extensions M of $\mathbb{Q}$ while $\mathrm{Gal}(\widetilde{M}/\mathbb{Q}) \cong D_4$ for only about 17%, Friedrichsen and Keliher prove the surprising result that for most imaginary quadratic fields, the D_4-quartic extensions dominate, and indeed can exceed the S_4-quartic extensions by an arbitrarily large factor. The asymptotic formulas quoted from [5] and [4] by Friedrichsen and Keliher involve the residue at $s = 1$ of the Dedekind zeta function of the quadratic extensions of F, and a large part of the work in [7] is devoted to estimating these residues.

The present note also involves D_4-extensions and makes reference to [5]. However the comparison is not with S_4-extensions but rather with other D_m-extensions, and instead of nonnormal quartic extensions the objects to be counted are dihedral Artin representations, an irreducible two-dimensional representation being called *dihedral* if its image is isomorphic to D_m for some $m \geqslant 3$. The leading coefficients in the two asymptotic formulas to be compared will turn out to differ only by an elementary factor, obviating the need for delicate estimates of residues of Dedekind zeta functions. But one of the two asymptotic formulas cannot simply be quoted from the literature, and most of this article is devoted to deriving it.

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To describe the contents of this note more systematically, we start with a simple group-theoretic fact: If $m \neq 4$ then the faithful irreducible representations of D_m can be monomially induced only from the unique cyclic subgroup of index 2 in D_m, but if $m = 4$ then they – or rather it, since up to isomorphism there is only one – can also be induced from either of the two noncyclic subgroups of index 2. Now fix a CM field K, let F be the maximal totally real subfield of K, and consider a dihedral Artin representation ρ of F induced from K. Let L be the fixed field of the kernel of ρ, so that ρ may be viewed as a faithful representation of $\mathrm{Gal}(L/F)$. If $\mathrm{Gal}(L/K)$ is the cyclic subgroup of index 2 in $\mathrm{Gal}(L/F)$ then we say that ρ is *canonically induced from* K, a condition which is automatically satisfied if the image of ρ is D_m with $m \neq 4$. But if $m = 4$ then it can happen that ρ is *noncanonically induced from* K, in other words, that $\mathrm{Gal}(L/K)$ is noncyclic. Let $\Delta^{\mathrm{can}}_{K/F}$ be the set of isomorphism classes of dihedral Artin representations of F which are canonically induced from K, and let $\Delta^{\mathrm{non}}_{K/F}$ be the corresonding set with "canonically induced" replaced by "noncanonically induced." Also, let $\delta^{\mathrm{can}}_{K/F}(x)$ and $\delta^{\mathrm{non}}_{K/F}(x)$ denote the number of elements of $\Delta^{\mathrm{can}}_{K/F}$ and $\Delta^{\mathrm{non}}_{K/F}$ respectively with conductor of absolute norm $\leqslant x$. Let d_F denote the discriminant of F and $d_{K/F}$ the absolute norm of the relative discriminant of K/F. Put $n = [F : \mathbb{Q}]$.

Theorem 1.

$$\delta^{\mathrm{can}}_{K/F}(x) \sim \frac{\sqrt{d_F/d_{K/F}}}{2\cdot(2\pi)^n}\cdot\frac{\mathrm{res}_{s=1}\zeta_K(s)}{\zeta_K(2)}\cdot x.$$

We emphasize that Theorem 1 applies only to quadratic CM extensions K of F, not to arbitrary quadratic extensions of F. This restriction is essential to our method, because we count canonically induced dihedral Artin representations by counting the inducing idele class characters, and such characters are subject to two conditions which mesh well only when K is CM: On the one hand, if K/F is any quadratic extension then an idele class character χ of K induces a dihedral representation of F canonically if and only if the restriction of χ to the ideles of F is trivial. On the other hand, the obstacle to constructing idele class characters of any sort is that they must be trivial on the principal ideles, and in particular – the crucial point – on the global units. In general this property is hard to achieve, but in the case of a CM extension K of F, and only in this case, the unit group of F is of finite index in the unit group of K. Thus the triviality of χ on the ideles of F, necessary to ensure a canonically induced dihedral representation, largely takes care of the required triviality of χ on the global units.

For any number field F, let $\delta_F(x)$ be the number of isomorphism classes of dihedral Artin representations ρ of F such that the absolute norm of the conductor of ρ is $\leqslant x$. The distinction between "canonical" and "noncanonical" induction now evaporates, because the distinction is not intrinsic to ρ: Every dihedral ρ is canonically induced from *some* quadratic extension of F. A glaring weakness in Theorem 1 is that it does not give an asymptotic formula for $\delta_F(x)$. Such a formula is known only for $F = \mathbb{Q}$, where Siegel's asymptotic averages of class numbers of primitive binary quadratic forms over $\mathbb{Z}$ give $\delta_{\mathbb{Q}}(x) \sim \pi x^{3/2}/(6\zeta(3))^2$ (see [14] and Theorem 2 of [11]). A crude heuristic based on Theorem 1 suggests that for an arbitrary totally real number field F, we have at least $\delta_F(x) \asymp x^{3/2}$ (sum over all quadratic CM extensions K of F, ignoring the difficulties). Admittedly, the heuristic takes account only of CM extensions F, but this limitation shouldn't matter: When $F = \mathbb{Q}$ the asymptotic for $\delta_{\mathbb{Q}}(x)$ is unchanged if we count only dihedral representations induced from imaginary quadratic fields.

Let us now return to the theme of comparing asymptotic formulas. We shall compare $\delta^{\mathrm{can}}_{K/F}(x)$ and $\delta^{\mathrm{non}}_{K/F}(x)$, but to put our comparison in perspective, take $F = \mathbb{Q}$ for a moment and consider all Artin representations of $\mathbb{Q}$ with image D_4, not just those noncanonically induced from a particular imaginary quadratic field K. Let $\delta_{\mathbb{Q},D_4}(x)$ be the number of isomorphism classes of all such representations with conductor $\leqslant x$. Then a recent result of Altuğ, Shankar, Varma, and Wilson [1] shows that $\delta_{\mathbb{Q},D_4}(x) \sim cx \log x$ for some $c > 0$. Thus the rate of growth is much lower than for $\delta_{\mathbb{Q}}(x)$. Nonetheless, by combining Theorem 1 with a result in [5], we obtain the following statement, where K is again an arbitrary CM field with maximal totally real subfield F:

Theorem 2.
$$\delta^{\mathrm{non}}_{K/F}(x) \sim \frac{\pi^n}{\sqrt{d_{K/F}d_F}}\,\delta^{\mathrm{can}}_{K/F}(x).$$

It follows that a positive proportion of dihedral Artin representations of F induced from K are noncanonically induced. In fact if $F = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{-d})$ with $d = 3, 4, 7$, or 8 then the number of isomorphism classes which are noncanonically induced from K is asymptotically greater than the total number of isomorphism classes which are canonically induced from K, including the canonically induced isomorphism classes with image D_4. But it must be added that if n is sufficiently large then $\pi^n < \sqrt{d_{K/F}d_F}$. This follows from various lower bounds for d_F in the literature. For example, Theorem 1 of Odlyzko [9] gives $d_F \geqslant 55^n$ if n is sufficiently large, whence $\pi^n/\sqrt{d_{K/F}d_F} \leqslant (\pi/\sqrt{55})^n < 1$. And even for small n, if F is fixed then there are at most finitely many K such that $\pi^n/\sqrt{d_{K/F}d_F} \geqslant 1$.

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1. Notation and conventions

Given a number field X, let $\mathcal{O}_X$ denotes its ring of integers, $\zeta_X(s)$ its Dedekind zeta function, $\mathbb{A}_X$ its ring of adeles, and $\mathbb{A}_X^\times$ its group of ideles. We write h_X and R_X for the class number and regulator of X and and w_X for the number of roots of unity in X. Also, if Y/X is an extension of number fields then $\mathfrak{d}_{Y/X}$ denotes the relative different ideal of Y/X, and we put $d_{Y/X} = \mathbf{N}\mathfrak{d}_{Y/X}$, writing simply $\mathfrak{d}_Y$ and d_Y if $X = \mathbb{Q}$. In particular, the well-known identity

$$d_Y = d_{Y/X} d_X^{[Y:X]} \tag{1}$$

follows from the relation $\mathfrak{d}_Y = \mathfrak{d}_{Y/X}\mathfrak{d}_X$ on taking absolute norm of both sides.

If $\mathfrak{q}$ is a nonzero integral ideal of X, then $C_X(\mathfrak{q})$ denotes the wide ray class group of X modulo $\mathfrak{q}$. By definition,

$$C_X(\mathfrak{q}) = I_X(\mathfrak{q})/P_{X,\mathfrak{q}}, \tag{2}$$

where $I_X(\mathfrak{q})$ is the multiplicative group of fractional ideals of X which are relatively prime to $\mathfrak{q}$ and $P_{X,\mathfrak{q}}$ is the subgroup of principal fractional ideals with a generator $\alpha \equiv 1 \bmod^* \mathfrak{q}$. The order of $C_X(\mathfrak{q})$ will be denoted $h_X(\mathfrak{q})$. Thus

$$h_X(\mathfrak{q}) = h_X \cdot \varphi_X(\mathfrak{q})/[\mathcal{O}_X^\times : \mathcal{O}_X^\times(\mathfrak{q})] \tag{3}$$

(cf. [8], p. 127), where $\varphi_X(\mathfrak{q}) = |(\mathcal{O}_X/\mathfrak{q})^\times|$ and $\mathcal{O}_X^\times(\mathfrak{q})$ is the subgroup of $\mathcal{O}_X^\times$ consisting of units $u \equiv 1 \bmod \mathfrak{q}$.

Write r_1 for the number of real embeddings $X \hookrightarrow \mathbb{R}$ of X and $2r_2$ for the number of nonreal complex embeddings $X \hookrightarrow \mathbb{C}$, so that $r_1 + 2r_2$ is the degree of X over $\mathbb{Q}$. If $r_1 > 0$ then one has the notion of the *narrow* ray class group of X modulo $\mathfrak{q}$, which we denote $C_X^{\mathrm{nar}}(\mathfrak{q})$. Let $P_{X,\mathfrak{q}}^{\mathrm{nar}}$ be the subgroup of $P_{X,\mathfrak{q}}$ consisting of principal fractional ideals with a generator $\alpha \equiv 1 \bmod^* \mathfrak{q}$ satifying $\sigma(\alpha) > 0$ for every embedding $\sigma : X \hookrightarrow \mathbb{R}$. Then $C_X^{\mathrm{nar}}(\mathfrak{q})$ is equal to the right-hand side of (2) with $P_{X,\mathfrak{q}}$ replaced by $P_{X,\mathfrak{q}}^{\mathrm{nar}}$. If $r_1 = 0$ then one can define the narrow ray class group of X modulo $\mathfrak{q}$ in the same way, but the group so defined is indistinguishable from the wide ray class group modulo $\mathfrak{q}$.

By an idele class character of X we mean as usual a continuous homomorphism $\chi : \mathbb{A}_X^\times \to \mathbb{C}^\times$ trivial on the diagonally embedded subgroup $X^\times$ of $\mathbb{A}_X^\times$. We write $\mathfrak{q}(\chi)$ for the conductor of χ and put $q(\chi) = \mathbf{N}\mathfrak{q}(\chi)$. If χ has finite order then it is canonically identified with a primitive character of $C_X^{\mathrm{nar}}(\mathfrak{q}(\chi))$, or simply of $C_X(\mathfrak{q}(\chi))$ if X is totally complex. The identification is encapsulated in the equation

$$\chi_v(\pi_v) = \chi(\mathfrak{p}_v),$$

in which χ has different meanings on the two sides of the equation: On the left-hand side, χ_v is the local component of the idele class character χ at a finite place v of X where χ is unramified, and π_v is a uniformizer of the completion X_v of X at v. On the right-hand side, χ is the corresponding primitive ray class character, and $\mathfrak{p}_v$ is the prime ideal of $\mathcal{O}_X$ corresponding to v.

Let $\overline{X}$ denote an algebraic closure of X. An Artin representation of X is a continuous homomorphism $\rho : \mathrm{Gal}(\overline{X}/X) \to \mathrm{GL}(V)$, where V is a finite-dimensional vector space over $\mathbb{C}$. In particular, a one-dimensional Artin representation of F can be viewed as a continuous homomorphism $\chi : \mathrm{Gal}(\overline{X}/X)^{\mathrm{ab}} \to \mathbb{C}^\times$, where $\mathrm{Gal}(\overline{X}/X)^{\mathrm{ab}}$ is the abelianization of $\mathrm{Gal}(\overline{X}/X)$. Via class field theory we identify such characters χ with idele class characters of X of finite order, or equivalently with primitive ray class characters.

We have already introduced the notation $\mathfrak{q}(\chi)$ for the conductor of an idele class character χ of X, and in view of the preceding remarks, the notation extends to one-dimensional Artin representations of X. More generally, if ρ is an Artin representation of X of any dimension then $\mathfrak{q}(\rho)$ denotes its conductor. We also put $q(\rho) = \mathbf{N}\mathfrak{q}(\rho)$ just as we put $q(\chi) = \mathbf{N}\mathfrak{q}(\chi)$ for an idele class character χ of X.

2. CM fields

Throughout the rest of this note, K is a CM field and F its maximal totally real subfield. Thus $K = F(\delta)$, where $\delta \notin F$ but δ^2 is a totally negative element of F. It follows that if τ is the nontrivial element of $\mathrm{Gal}(K/F)$ and $z = x + \delta y$ with $x, y \in F$, then for any embedding $\sigma : K \hookrightarrow \mathbb{C}$,

$$\sigma(\tau(z)) = \sigma(x) - \sigma(\delta)\sigma(y) = \overline{\sigma(z)}.$$

Thus while there is no canonical embedding of K into $\mathbb{C}$, there is also no ambiguity in writing $\tau(z)$ simply as $\overline{z}$.

A standard invariant of K as a CM field is the group index

$$Q_K = [\mathcal{O}_K^\times : \mu_K \mathcal{O}_F^\times],$$

where μ_K is the group of roots of unity in K. If $u \in \mathcal{O}_K^\times$ then $u/\overline{u}$ is a unit with absolute value 1 in every embedding $K \hookrightarrow \mathbb{C}$; hence $u/\overline{u} \in \mu_K$. It is a well-known and easily verified remark that the map from $\mathcal{O}_K^\times/(\mu_K\mathcal{O}_F^\times)$ to μ_K/μ_K^2 sending the coset of u to the coset of $u/\overline{u}$ is injective, whence Q_K is 1 or 2.

Let $\mathfrak{q}$ be a nonzero ideal of $\mathcal{O}_F$. The map $\mathfrak{a} \mapsto \mathfrak{a}\mathcal{O}_K$ from $I_F(\mathfrak{q})$ to $I_K(\mathfrak{q}\mathcal{O}_K)$ sends $P_{F,\mathfrak{q}}$ to $P_{K,\mathfrak{q}\mathcal{O}_K}$ and so gives rise to a map $C_F(\mathfrak{q}) \to C_K(\mathfrak{q}\mathcal{O}_K)$ which we shall refer to as the conorm (on ray class groups). In the same way we could define a conorm from $C_F^{\mathrm{nar}}(\mathfrak{q})$ to $C_K(\mathfrak{q})$, but since the latter conorm would factor through the quotient map from $C_F^{\mathrm{nar}}(\mathfrak{q})$ to $C_F(\mathfrak{q})$, its image in $C_K(\mathfrak{q}\mathcal{O}_K)$ would be the same, and it is precisely the size of the image which is our primary concern.

Proposition 1. *For all but finitely many nonzero ideals $\mathfrak{q}$ of $\mathcal{O}_K$, the conorm*

$$C_F(\mathfrak{q}) \to C_K(\mathfrak{q}\mathcal{O}_K)$$

is injective. In fact let $m \geqslant 2$ be the largest integer such that the m*th roots of unity are contained in K. If the conorm is not injective then $\mathfrak{q}$ divides $m\mathcal{O}_F$.*

Proof. Suppose that there is an element $c \neq 1$ in the kernel of the conorm, and choose an integral ideal $\mathfrak{a} \in c$. Then $\mathfrak{a}\mathcal{O}_K$ is generated by an element $\alpha \equiv 1 \bmod \mathfrak{q}\mathcal{O}_K$. As $\mathfrak{a}\mathcal{O}_K = \alpha\mathcal{O}_K$ it follows that $\alpha/\overline{\alpha}$ is a unit of $\mathcal{O}_K$ and in fact a root of unity, because $|\sigma(\alpha/\overline{\alpha})| = 1$ for every embedding $\sigma : K \hookrightarrow \mathbb{C}$. Thus

$$\alpha = \zeta\,\overline{\alpha}, \tag{4}$$

where ζ is a root of unity. Then $\zeta \neq 1$, else $\alpha \in F$ and $c = 1$, a contradiction. On the other hand, $\alpha \equiv 1 \bmod \mathfrak{q}\mathcal{O}_K$ and consequently $\overline{\alpha} \equiv 1 \bmod \mathfrak{q}\mathcal{O}_K$ also, whence $\zeta \equiv 1 \bmod \mathfrak{q}\mathcal{O}_K$ by (4). In other words, $\mathfrak{q}$ divides $1-\zeta$. Now let ξ denote a primitive mth root of unity in K. Then $1-\zeta$ is one of the factors on the left-hand side of the equation

$$\prod_{j=1}^{m-1}(1-\xi^j) = m,$$

whence $\mathfrak{q}$ divides m. □

3. Kef characters

We remind the reader that K always denotes a CM field and F its maximal totally real subfield. By a K/F-character we mean an idele class character of K of finite order which is trivial on the ideles of F. If K is imaginary quadratic then the classical name for a K/F-character is *ring class character*, and if χ factors through $\mathrm{Gal}(K_\infty/K)$ for some $\mathbb{Z}_p$-extension K_∞ of K then the term *anticyclotomic character* is also in use, although the appropriateness of this designation really depends on the validity of Leopoldt's conjecture for F. A K/F-character could also be called a *kef character*, "kef" being a synonym for "bliss" and an acronym for "K effacing F." Whatever the terminology, the defining property of such finite-order characters χ is that $\chi|\mathbb{A}_F^\times = 1$.

Like any idele class character of K of finite order, a K/F-character can be identified with a character $\chi : \mathrm{Gal}(\overline{K}/K)^{\mathrm{ab}} \to \mathbb{C}^\times$. The identity $\chi|\mathbb{A}_F^\times = 1$ then becomes $\chi \circ \mathrm{tran}_{K/F} = 1$, where $\mathrm{tran}_{K/F} : \mathrm{Gal}(\overline{F}/F)^{\mathrm{ab}} \to \mathrm{Gal}(\overline{K}/K)^{\mathrm{ab}}$ is the transfer from $\mathrm{Gal}(\overline{F}/F)$ to $\mathrm{Gal}(\overline{K}/K)$ and we take $\overline{K} = \overline{F}$. Thus in the Galois framework a K/F-character is a character of $\mathrm{Gal}(\overline{K}/K)^{\mathrm{ab}}$ trivial on the image of the transfer from $\mathrm{Gal}(\overline{F}/F)^{\mathrm{ab}}$. There is also a characterization in terms of primitive ray class characters, but it depends on the following simple remark.

Proposition 2. *If χ is a K/F-character then $\mathfrak{q}(\chi) = \mathfrak{q}\mathcal{O}_K$ for some nonzero ideal $\mathfrak{q}$ of $\mathcal{O}_F$.*

Proof. This is routine and can also be deduced from a general theorem of Serre [13], but we briefly recall the argument. Write $a(\chi_w)$ for the conductor-exponent of the component of χ at a finite place w of K, and let v be the place of F below w. There are two points to be verified:

- If v splits in K, so that there is a second place $\overline{w}$ above v, then $a(\chi_w) = a(\chi_{\overline{w}})$.
- If v ramifies in K then $a(\chi_w)$ is even.

In the case where v splits, observe that the condition $\chi|\mathbb{A}_F^\times = 1$ implies that for $z \in F_v^\times$ we have $\chi_w(z)\chi_{\overline{w}}(z) = 1$. As K_w and $K_{\overline{w}}$ can be identified with F_v and hence with each other, it is meaningful to write $\chi_w = \chi_{\overline{w}}^{-1}$, and consequently $a(\chi_w) = a(\chi_{\overline{w}})$.

In the case where v ramifies, let $\pi \in F_v$ be a uniformizer of the ring of integers $\mathcal{O}_v$ of F_v. Suppose that $a(\chi_w)$ is an odd integer n. Then $\pi^{(n-1)/2}$ has w-valuation $n-1$, and consequently $\chi_w(1 + x\pi^{(n-1)/2}) \neq 1$ for some $x \in \mathcal{O}_v$, contradicting the fact that $\chi|\mathbb{A}_F^\times = 1$. $\square$

We can now transfer the previous discussion to ray class groups. Let $\mathfrak{q}$ be a nonzero ideal of $\mathcal{O}_F$. We say that a primitive ray class character of K of conductor $\mathfrak{q}\mathcal{O}_K$ is a K/F-character if it is trivial on the image of the conorm from $C_F(\mathfrak{q})$ to $C_K(\mathfrak{q}\mathcal{O}_K)$. Equivalently, we could demand triviality on the image of the conorm from $C_F^{\mathrm{nar}}(\mathfrak{q})$ to $C_K(\mathfrak{q}\mathcal{O}_K)$, because as previously noted, the latter conorm has the same image. The main point is that the condition for a primitive ray class character to be a K/F-character matches the condition already given for idele class characters.

The next observation is that the K/F-characters of conductor dividing $\mathfrak{q}\mathcal{O}_K$ are in one-to-one correspondence with the characters of the quotient group

$$C_{K/F}(\mathfrak{q}) = C_K(\mathfrak{q}\mathcal{O}_K)/(\text{image of } C_F(\mathfrak{q})). \tag{5}$$

Let $h_{K/F}(\mathfrak{q})$ be the order of $C_{K/F}(\mathfrak{q})$. Then $h_{K/F}(\mathfrak{q})$ is also the number of K/F-characters of conductor dividing $\mathfrak{q}\mathcal{O}_K$.

Proposition 3. *For all but finitely many nonzero ideals $\mathfrak{q}$ of $\mathcal{O}_F$,*

$$h_{K/F}(\mathfrak{q}) = \frac{h_{K/F} \cdot \varphi_{K/F}(\mathfrak{q})}{u_{K/F}},$$

where $h_{K/F} = h_K/h_F$, $u_{K/F} = [\mathcal{O}_K^\times : \mathcal{O}_F^\times]$, and $\varphi_{K/F}(\mathfrak{q}) = \varphi_K(\mathfrak{q}\mathcal{O}_K)/\varphi_F(\mathfrak{q})$.

Proof. We have seen (Proposition 1) that for all but finitely many $\mathfrak{q}$ the conorm from $C_F(\mathfrak{q})$ to $C_K(\mathfrak{q}\mathcal{O}_K)$ is injective, and when the conorm is injective we can combine (5) and (3) to obtain

$$h_{K/F}(\mathfrak{q}) = h_{K/F} \cdot \varphi_{K/F}(\mathfrak{q}) \cdot \frac{[\mathcal{O}_F^\times : \mathcal{O}_F^\times(\mathfrak{q})]}{[\mathcal{O}_K^\times : \mathcal{O}_K^\times(\mathfrak{q}\mathcal{O}_K)]}.$$

Thus it suffices to see that

$$u_{K/F} = \frac{[\mathcal{O}_K^\times : \mathcal{O}_K^\times(\mathfrak{q}\mathcal{O}_K)]}{[\mathcal{O}_F^\times : \mathcal{O}_F^\times(\mathfrak{q})]} \tag{6}$$

for all but finitely many $\mathfrak{q}$.

A sufficient condition for the validity of (6) is

$$\mathcal{O}_F^\times(\mathfrak{q}) = \mathcal{O}_K^\times(\mathfrak{q}\mathcal{O}_K), \tag{7}$$

for (7) implies that the natural map

$$\mathcal{O}_F^\times/\mathcal{O}_F^\times(\mathfrak{q}) \to \mathcal{O}_K^\times/\mathcal{O}_K^\times(\mathfrak{q}\mathcal{O}_K)$$

is injective with cokernel $\mathcal{O}_K^\times/\mathcal{O}_F^\times$. Suppose that contrary to (7), there exists $u \in \mathcal{O}_K^\times(\mathfrak{q}\mathcal{O}_K)$ such that $u \neq \overline{u}$. Then $u/\overline{u}$ is a root of unity $\zeta \neq 1$, and reprising an argument in the proof of Proposition 1, we see that $\mathfrak{q}$ divides $1-\zeta$ and therefore also m, where m is the largest integer $\geqslant 2$ such that K contains μ_m. Hence the set of $\mathfrak{q}$ such that (7) fails is finite. □

4. An asymptotic formula

Let $\vartheta_{K/F}(x)$ be the number of K/F-characters χ with $q(\chi) \leqslant x$. Our goal is an asymptotic formula for $\vartheta_{K/F}(x)$. As we have already remarked, $h_{K/F}(\mathfrak{q})$ is the number of K/F-characters of conductor *dividing* $\mathfrak{q}\mathcal{O}_K$. Let $h^*_{K/F}(\mathfrak{q})$ be the number of K/F-characters of conductor *precisely* $\mathfrak{q}\mathcal{O}_K$. Then $h_{K/F}(\mathfrak{q})$ and $h^*_{K/F}(\mathfrak{q})$ are related by the equations

$$h_{K/F}(\mathfrak{q}) = \sum_{\mathfrak{r}|\mathfrak{q}} h^*_{K/F}(\mathfrak{r}) \tag{8}$$

and

$$h^*_{K/F}(\mathfrak{q}) = \sum_{\mathfrak{r}|\mathfrak{q}} h_{K/F}(\mathfrak{q})\mu_F(\mathfrak{r}), \tag{9}$$

where $\mathfrak{r}$ runs over ideals of $\mathcal{O}_F$ dividing $\mathfrak{q}$ and μ_F is the Möbius function of F, defined by

$$\zeta_F(s)^{-1} = \sum_{\mathfrak{r}} \mu_F(\mathfrak{r})(\mathbf{N}\mathfrak{r})^{-s}.$$

But the main point is that $\vartheta_{K/F}(x)$ can be expressed in terms of $h^*_{K/F}(\mathfrak{q})$: Since $\mathbf{N}(\mathfrak{q}\mathcal{O}_K) = (\mathbf{N}\mathfrak{q})^2$, we have

$$\vartheta_{K/F}(x) = \sum_{(\mathbf{N}\mathfrak{q})^2 \leqslant x} h^*_{K/F}(\mathfrak{q}), \tag{10}$$

where the sum runs over nonzero ideals $\mathfrak{q}$ of $\mathcal{O}_F$ with absolute norm $\leqslant \sqrt{x}$.

Next consider the formal Dirichlet series

$$H_{K/F}(s) = \sum_{\mathfrak{q}} h_{K/F}(\mathfrak{q})(\mathbf{N}\mathfrak{q})^{-2s} \tag{11}$$

and

$$H^*_{K/F}(s) = \sum_{\mathfrak{q}} h^*_{K/F}(\mathfrak{q})(\mathbf{N}\mathfrak{q})^{-2s}, \tag{12}$$

which by virtue of (8) and (9) satisfy

$$H_{K/F}(s) = \zeta_F(2s)H^*_{K/F}(s). \tag{13}$$

We shall rewrite (11) and (12) using Proposition 3.

We begin with the tautology

$$H_{K/F}(s) = \mathcal{H}(s) + \mathcal{E}(s), \tag{14}$$

where in the notation of Proposition 3,

$$\mathcal{H}(s) = (h_{K/F}/u_{K/F}) \sum_{\mathfrak{q}} \varphi_{K/F}(\mathfrak{q})(\mathbf{N}\mathfrak{q})^{-2s}$$

and

$$\mathcal{E}(s) = \sum_{\mathfrak{q}} (h_{K/F}(\mathfrak{q}) - (h_{K/F}/u_{K/F})\varphi_{K/F}(\mathfrak{q}))(\mathbf{N}\mathfrak{q})^{-2s}$$

(for notational simplicity we do not append a subscript K/F on $\mathcal{H}$ and $\mathcal{E}$). The significance of (14) is that $\mathcal{E}(s)$ is a finite Dirichlet series by Proposition 3 while $\mathcal{H}(s)$ has an Euler product, as we shall now see.

By the Chinese remainder theorem, both $\mathfrak{q} \mapsto \varphi_F(\mathfrak{q})$ and $\mathfrak{q} \mapsto \varphi_K(\mathfrak{q}\mathcal{O}_K)$ are multiplicative functions of $\mathfrak{q}$, so $\mathfrak{q} \mapsto \varphi_{K/F}(\mathfrak{q})$ is also. Hence

$$\mathcal{H}(s) = (h_{K/F}/u_{K/F}) \prod_{\mathfrak{p}} (1 + \sum_{k \geqslant 1} \varphi_{K/F}(\mathfrak{p}^k)(\mathbf{N}\mathfrak{p})^{-2ks}), \tag{15}$$

where $\mathfrak{p}$ runs over nonzero prime ideals of $\mathcal{O}_F$. Let κ be the primitive quadratic ray class character of F corresponding to the extension K/F. By considering cases according as $\mathfrak{p}$ splits, remains prime, or ramifies, we find that

$$\varphi_{K/F}(\mathfrak{p}^k) = \frac{|(\mathcal{O}_K/\mathfrak{p}^k\mathcal{O}_K)^\times|}{|(\mathcal{O}_F/\mathfrak{p}^k)^\times|} = (\mathbf{N}\mathfrak{p} - \kappa(\mathfrak{p}))(\mathbf{N}\mathfrak{p})^{k-1}.$$

Thus (15) can be rewritten:

$$\mathcal{H}(s) = (h_{K/F}/u_{K/F}) \prod_{\mathfrak{p}} (1 + (1 - \kappa(\mathfrak{p})/\mathbf{N}\mathfrak{p}) \sum_{k \geqslant 1} (\mathbf{N}\mathfrak{p})^{k(1-2s)})$$

Since

$$1 + (1 - \kappa(\mathfrak{p})/\mathbf{N}\mathfrak{p}) \frac{(\mathbf{N}\mathfrak{p})^{(1-2s)}}{1 - (\mathbf{N}\mathfrak{p})^{(1-2s)}} = \frac{1 - \kappa(\mathfrak{p})(\mathbf{N}\mathfrak{p})^{-2s}}{1 - (\mathbf{N}\mathfrak{p})^{1-2s}},$$

our expression for $\mathcal{H}(s)$ simplifies:

$$\mathcal{H}(s) = (h_{K/F}/u_{K/F})\zeta_F(2s-1)/L(2s, \kappa).$$

Finally, returning to (14) and then to (13), we have

$$H_{K/F}(s) = (h_{K/F}/u_{K/F})\zeta_F(2s-1)/L(2s, \kappa) + \mathcal{E}(s) \tag{16}$$

and

$$H^*_{K/F}(s) = (h_{K/F}/u_{K/F})\zeta_F(2s-1)/\zeta_K(2s) + \mathcal{E}(s)/\zeta_F(2s) \tag{17}$$

after using the factorization $\zeta_K(s) = \zeta_F(s)L(s, \kappa)$.

While $H_{K/F}(s)$ and $H^*_{K/F}(s)$ were originally introduced in (11) and (12) as formal Dirichlet series with nonnegative coefficients, we see from (16) and (17) that these Dirichlet series are absolutely convergent for $\Re(s) > 1$. Furthermore, since $\mathcal{E}(s)$ is a finite Dirichlet series and therefore entire, it also follows that $H_{K/F}(s)$ and $H^*_{K/F}(s)$ are holomorphic for $\Re(s) > 1/2$ except for simple poles at $s = 1$. We shall compute the residue of $H^*_{K/F}(s)$ at $s = 1$.

Proposition 4. *Put $n = [F : \mathbb{Q}]$. Then*

$$\mathrm{res}_{s=1} H^*_{K/F}(s) = \frac{\sqrt{d_{K/F} d_F}}{(2\pi)^n} \cdot \frac{\mathrm{res}_{s=1}\zeta_K(s)}{\zeta_K(2)}$$

Proof. The residue of $\zeta_F(2s-1)$ at $s=1$ is half the residue of $\zeta_F(s)$. Hence the class number formula gives

$$\frac{\mathrm{res}_{s=1}\zeta_F(2s-1)}{\mathrm{res}_{s=1}\zeta_K(s)} = \frac{2^{n-1}h_F R_F/(w_F\sqrt{d_F})}{(2\pi)^n h_K R_K/(w_K\sqrt{d_K})}. \tag{18}$$

Now if $|\ast|_1, |\ast|_2, \ldots, |\ast|_{n-1}$ are $n-1$ of the n archimedean absolute values of F and $\epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1}$ are coset representatives modulo $\{\pm 1\}$ of a basis for $\mathcal{O}_F^\times/\{\pm 1\}$ then R_F is the absolute value of the determinant of the $(n-1)\times(n-1)$ regulator matrix:

$$R_F = |\det(\log|\epsilon_j|_k)|$$

$(1 \leqslant j, k \leqslant n-1)$. To compute R_K one multiplies each entry $\log|\epsilon_j|_k$ by 2 and then divides the determinant by Q. Hence $R_K = 2^{n-1}R_F/Q$, and (18) becomes

$$\frac{\mathrm{res}_{s=1}\zeta_F(2s-1)}{\mathrm{res}_{s=1}\zeta_K(s)} = \frac{Q(w_K/w_F)(\sqrt{d_K}/\sqrt{d_F})}{h_{K/F}(2\pi)^n}.$$

But $d_K = d_{K/F}d_F^2$ by (1) and $u_{K/F} = Q(w_K/w_F)$, so we get

$$(h_{K/F}/u_{K/F})\mathrm{res}_{s=1}\zeta_F(2s-1) = \frac{\sqrt{d_{K/F}d_F}}{(2\pi)^n}\mathrm{res}_{s=1}\zeta_K(s).$$

Now the stated formula follows from (17). □

Recognizing from (10) and (12) that $\vartheta_{K/F}(x)$ is the summatory function of $H^*_{K/F}(s)$, and applying a standard tauberian theorem (e. g. Theorem 7.7 on p. 154 of [2]), we obtain:

Theorem 3.

$$\vartheta_{K/F}(x) \sim \frac{\sqrt{d_{K/F}d_F}}{(2\pi)^n}\cdot\frac{\mathrm{res}_{s=1}\zeta_K(s)}{\zeta_K(2)}\cdot x.$$

5. Artin representations

As usual, we say that an idele class character ν of a number field X is *quadratic* if $\nu^2 = 1$ but $\nu \neq 1$. We write $\lambda_X(x)$ for the number of quadratic idele class characters ν of X with $q(\nu) \leqslant x$. Equivalently, $\lambda_X(x)$ is the number of quadratic extensions Y of X in some fixed algebraic closure $\overline{X}$ such that $d_{Y/X} \leqslant x$. Or we could define $\lambda_X(x)$ to be the number of quadratic extensions Y of X such that $\mathbf{N}\mathcal{D}_{Y/X} \leqslant x$, where $\mathcal{D}_{Y/X}$ is the discriminant ideal of Y/X. Of course the equivalence of the three definitions follows from the fact that if ν corresponds to Y then

$$\mathfrak{q}(\nu) = \mathcal{D}_{Y/X} = N_{Y/X}(\mathfrak{d}_{Y/X}),$$

where $N_{Y/X}$ is the norm map on ideals. We shall think of $\lambda_X(x)$ as counting quadratic characters, but the other interpretations put the asymptotic formulas of Datskovsky and Wright [6] and of Cohen, Diaz y Diaz, and Olivier [5] at our disposal.

The version proved in [5] is particularly convenient for our purposes. In the case of a CM field K of degree $2n$ it takes the form

$$\lambda_K(x) \sim 2^{-n}\cdot\frac{\mathrm{res}_{s=1}\zeta_K(s)}{\zeta_K(2)}\cdot x \tag{19}$$

(Corollary 1.2 of [5]). Thus in light of Theorem 3 we have:

Theorem 4.
$$\lambda_K(x) \sim \frac{\pi^n}{\sqrt{d_{K/F}d_F}}\,\vartheta_{K/F}(x).$$

However Theorems 3 and 4 do not yet refer to dihedral Artin representations. To move in that direction, we let $\lambda_{K/F}(x)$ be the number of quadratic K/F-characters ν with $q(\nu) \leqslant x$, and we introduce the functions

$$\vartheta(x) = (\vartheta_{K/F}(x) - \lambda_{K/F}(x) - 1)/2 \tag{20}$$

and

$$\lambda(x) = (\lambda_K(x) - \lambda_{K/F}(x))/2. \tag{21}$$

The function $\lambda_{K/F}(x)$ is from our point of view an error term, and $\vartheta(x)$ and $\lambda(x)$ are closely related to the functions $\delta^{\mathrm{can}}_{K/F}(x)$ and $\delta^{\mathrm{non}}_{K/F}(x)$ of the introduction. In fact we shall prove:

Proposition 5. $\lambda_{K/F}(x) = O(\sqrt{x})$.

Proposition 6. $\delta^{\mathrm{can}}_{K/F}(x) = \vartheta(x/d_{K/F})$.

Proposition 7. $\delta^{\mathrm{non}}_{K/F}(x) = \lambda(x/d_{K/F})$.

Granting these three propositions, let us deduce Theorems 1 and 2. Inserting (20) in Theorem 3 and applying Proposition 5, we see that

$$\vartheta(x) \sim \frac{\sqrt{d_{K/F}d_F}}{2\cdot(2\pi)^n}\cdot\frac{\mathrm{res}_{s=1}\zeta_K(s)}{\zeta_K(2)}\cdot x. \tag{22}$$

Theorem 1 now follows from (22) and Proposition 6. To obtain Theorem 2, we insert (20) and (21) in Theorem 4 and apply Proposition 5. The result is

$$\lambda(x) \sim \frac{\pi^n}{\sqrt{d_{K/F}d_F}}\,\vartheta(x). \tag{23}$$

Theorem 2 follows from (23) and Propositions 6 and 7.

It remains to prove Propositions 5, 6, and 7.

6. Proof of Proposition 5

Let $N_{K/F} : \mathbb{A}_K^\times \to \mathbb{A}_F^\times$ be the adelic norm. Also, write κ for the quadratic idele class character of F corresponding to the extension K/F. The proof of Proposition 5 is based on the following remark.

Proposition 8. *An idele class character χ of K is a quadratic K/F-character if and only if $\chi = \psi \circ N_{K/F}$ for some quadratic idele class character ψ of F with $\psi \neq \kappa$.*

Proof. This is standard, at least when K is imaginary quadratic (cf. [12], §7). The argument in general is the same, but we nonetheless recall it.

Suppose first that $\chi = \psi \circ N_{K/F}$ with ψ as stated. Then $\chi \neq 1$ because $\psi \neq \kappa$. Therefore χ is quadratic. Now if $a \in \mathbb{A}_F^\times$ then $N_{K/F}(a) = a^2$, so $\chi(a) = \psi(a^2) = \psi(a)^2 = 1$. Hence $\chi|\mathbb{A}_F^\times = 1$ and χ is a K/F-character.

Conversely, suppose that χ is a quadratic K/F-character. Since $\chi|\mathbb{A}_F^\times = 1$ we have $\chi \circ N_{K/F} = 1$, and of course $\chi|K^\times = 1$ also. Thus viewing χ as a character of $\mathbb{A}^\times/K^\times$, we have $\chi(a^{\tau+1}) = 1$ for all $a \in \mathbb{A}_K^\times/K^\times$, where τ denotes the automorphism of $\mathbb{A}_K^\times/K^\times$ corresponding to the nontrivial element of $\mathrm{Gal}(K/F)$. But since χ is quadratic we also have $\chi(a^2)=1$, whence $\chi(a^{\tau-1})=1$. By class field theory (specifically, the vanishing of the Galois cohomology group $H^1(\mathbb{A}_K^\times/K^\times)$) it follows that χ is trivial on the kernel of $N_{K/F}$, now viewed as a map from $\mathbb{A}_K^\times/K^\times$ to $\mathbb{A}_F^\times/F^\times$. Therefore χ factors through $N_{K/F}$ to give a character ψ on the image of $N_{K/F}$ in $\mathbb{A}_F^\times/F^\times$ such that $\chi=\psi\circ N_{K/F}$. The image of $N_{K/F}$ has index 2 in $\mathbb{A}_F^\times/F^\times$, and extending ψ to $\mathbb{A}_F^\times/F^\times$, we obtain an idele class character of F, which we will also denote ψ. If $a\in\mathbb{A}_F^\times$ then $1=\chi(a)=\psi(a^2)=\psi(a)^2$, so ψ is quadratic. □

Given a quadratic K/F-character χ, let ψ be a quadratic idele class character of F such that $\chi=\psi\circ N_{K/F}$. The existence of ψ follows from Proposition 8, and we now claim that

$$q(\psi)\leqslant c\sqrt{q(\chi)} \tag{24}$$

with a constant c depending only on K. Granting the claim, we have $\lambda_{K/F}(x)\leqslant\lambda_F(c\sqrt{x})$, a bound which in combination with the asymptotic

$$\lambda_F(x)\sim\frac{\mathrm{res}_{s=1}\zeta_F(s)}{\zeta_F(2)}\cdot x$$

(Corollary 1.2 of [5]) gives Proposition 5. So to complete the proof of the latter proposition it suffices to verify (24).

Put $c=8^n d_{K/F}$, and let $\mathfrak{q}(\psi)=\mathfrak{t}\cdot\mathfrak{r}\cdot\mathfrak{u}$ be the unique decomposition of $\mathfrak{q}(\psi)$ as a product of ideals $\mathfrak{t}$, $\mathfrak{r}$, and $\mathfrak{u}$ of $\mathcal{O}_F$ satisfying the following conditions:

(i) The prime ideals dividing $\mathfrak{t}$ have residue characteristic 2.
(ii) The prime ideals dividing $\mathfrak{r}$ have odd residue characteristic and are ramified in K.
(iii) The prime ideals dividing $\mathfrak{u}$ have odd residue characteristic and are unramified in K.

To verify (24), it suffices to prove the following assertions:

(i) $\mathfrak{t}$ divides $8\mathcal{O}_F$.
(ii) $\mathfrak{r}$ divides $\mathcal{D}_{K/F}$, the discriminant ideal of K/F.
(iii) $\mathfrak{u}\mathcal{O}_K$ divides $\mathfrak{q}(\chi)$.

Indeed (i), (ii), and (iii) imply respectively that $\mathbf{N}\mathfrak{t}$ divides 8^n, that $\mathbf{N}\mathfrak{r}$ divides $d_{K/F}$, and that $(\mathbf{N}\mathfrak{u})^2$ divides $q(\chi)$, whence

$$q(\psi)=(\mathbf{N}\mathfrak{t})(\mathbf{N}\mathfrak{r})(\mathbf{N}\mathfrak{u})\leqslant c\sqrt{q(\chi)},$$

proving (24).

To prove (i), let $\mathfrak{p}$ be a prime dividing $\mathfrak{t}$. It is a standard remark that the multiplicity of a prime ideal $\mathfrak{p}$ of residue characteristic 2 in the conductor of any quadratic character of F is $\leqslant 2e(p)+1$, where $e(p)$ is the ramification index of $\mathfrak{p}$ over $2\in\mathbb{Q}$. Thus $\mathfrak{t}$ divides the ideal

$$\mathfrak{t}'=\prod_{\mathfrak{p}|\mathfrak{t}}\mathfrak{p}^{2e(p)+1}.$$

But $\mathfrak{t}'$ divides the ideal

$$\prod_{\mathfrak{p}|2\mathcal{O}_F}\mathfrak{p}^{3e(p)}=(2\mathcal{O}_F)^3,$$

in other words $8\mathcal{O}_F$.

To prove (ii), let $\mathfrak{p}$ be a prime dividing $\mathfrak{r}$. Since $\mathfrak{p}$ has odd residue characteristic and ψ is quadratic, the multiplicity of $\mathfrak{p}$ in $\mathfrak{q}(\psi)$, and hence in $\mathfrak{r}$, is 1. On the other hand, since $\mathfrak{p}$ is ramified in K it divides $\mathcal{D}_{K/F}$. Since $\mathfrak{p}$ is an arbitrary prime divisor of $\mathfrak{r}$, it follows that $\mathfrak{r}$ divides $\mathcal{D}_{K/F}$.

To prove (iii), let $\mathfrak{p}$ be a prime dividing $\mathfrak{u}$, let $\mathfrak{P}$ be a prime of K lying above $\mathfrak{p}$, let $F_\mathfrak{p}$ and $K_\mathfrak{P}$ be the respective completions, and let $\mathcal{O}_\mathfrak{p}$ and $\mathcal{O}_\mathfrak{P}$ be their rings of integers. Since $K_\mathfrak{P}$ is unramified over $F_\mathfrak{p}$, the norm from $\mathcal{O}_\mathfrak{P}^\times$ to $\mathcal{O}_\mathfrak{p}^\times$ is surjective, whence the local component at $\mathfrak{P}$ of $\chi = \psi \circ N_{K/F}$ is ramified, because ψ is ramified at $\mathfrak{p}$. So $\mathfrak{P}$ divides $\mathfrak{q}(\chi)$. On the other hand, the multiplicity of $\mathfrak{p}$ in $\mathfrak{u}$ is 1, because ψ is quadratic and $\mathfrak{p}$ has odd residue characteristic. As $\mathfrak{u}\mathcal{O}_K$ is just the product of the prime ideals $\mathfrak{P}$ lying above the prime divisors $\mathfrak{p}$ of $\mathfrak{u}$, it divides $\mathfrak{q}(\chi)$.

7. Proof of Proposition 6

The expresssion $\vartheta_{K/F}(x) - \lambda_{K/F}(x) - 1$ is the number of K/F-characters χ of order $\geqslant 3$ with $q(\chi) \leqslant x$. Hence $\vartheta(x)$ is the number of unordered pairs $\{\chi, \chi^{-1}\}$ where each such character is paired with its complex conjugate. And $\vartheta(x/d_{K/F})$ is the number of such unordered pairs with $d_{K/F}q(\chi) \leqslant x$. Let ρ be the Artin representation of F induced by χ; we write $\rho = \mathrm{ind}_{K/F}\chi$. By the conductor-discriminant formula, $q(\rho) = d_{K/F}q(\chi)$.

Write τ for the nontrivial element of $\mathrm{Gal}(K/F)$ and also for the automorphism of $\mathbb{A}_K^\times$ it affords. The fact that $\chi(a^{1+\tau}) = 1$ for $a \in \mathbb{A}_K^\times$ means that $\chi(a^\tau) = \chi^{-1}(a)$, and consequently (after viewing χ as a character of $\mathrm{Gal}(\overline{F}/K)$ and lifting τ to $\mathrm{Gal}(\overline{F}/F)$) that $\chi(\tau g \tau^{-1}) = \chi^{-1}(g)$ for $g \in \mathrm{Gal}(\overline{K}/K)$. Thus $\mathrm{ind}_{K/F}\chi = \mathrm{ind}_{K/F}\chi^{-1}$. If we put $\rho = \mathrm{ind}_{K/F}\chi$ as before and let L be the fixed field of the kernel of ρ then $[L : F] = 2m$, where $m \geqslant 3$ is the order of χ. Thus $\mathrm{Gal}(L/K)$ is cyclic of index 2 in $\mathrm{Gal}(L/F)$.

To conclude that ρ is a dihedral Artin representation of F canonically induced from K, recall that the condition $\chi|\mathbb{A}_F^\times = 1$ for idele class characters is equivalent to the condition $\chi \circ \mathrm{tran}_{K/F} = 1$ for characters of $\mathrm{Gal}(\overline{F}/F)$. Now it is a standard remark (cf. Propositions 5.3 and 5.4 of [10]; I learned these facts from lectures of Serre long ago) that the condition $\chi \circ \mathrm{tran}_{K/F} = 1$, or in other words the condition that χ be a K/F-character, is precisely the condition for $\rho = \mathrm{ind}_{K/F}\chi$ to be dihedral. Furthermore, by Frobenius reciprocity χ and χ^{-1} are the only characters of $\mathrm{Gal}(\overline{K}/K)$ inducing the isomorphism class of ρ. So we obtain a bijection between the unordered pairs $\{\chi, \chi^{-1}\}$ counted by $\vartheta(x/d_{K/F})$ and the isomorphism classes of dihedral Artin representations counted by $\delta_{K/F}^{\mathrm{can}}(x)$.

8. Proof of Proposition 7

The first point is that if Q is a quadratic extension of K which is Galois over F then $\mathrm{Gal}(Q/F) \cong (\mathbb{Z}/2\mathbb{Z})^2$. Indeed if $\tau \in \mathrm{Gal}(Q/F)$ is any complex conjugation corresponding to an archimedean embedding of Q then $\{\tau, 1\}$ is a subgroup of $\mathrm{Gal}(Q/F)$ which is complementary to $\mathrm{Gal}(Q/K)$.

Thus if Q is Galois over F then there is a quadratic extension R of F such that Q is the compositum of K and R. Let ψ be the quadratic idele class character of F corresponding to the extension R. Then $\psi \circ N_{K/F}$ is the quadratic idele class character of K corresponding to Q, and from Proposition 8 we see that $\psi \circ N_{K/F}$ is a quadratic K/F-character. Conversely, Proposition 8 also shows that if χ is a quadratic K/F-character then $\chi = \psi \circ N_{K/F}$ for some ψ as above, and therefore the extension Q of K corresponding to χ is Galois over F with Galois group $(\mathbb{Z}/2\mathbb{Z})^2$.

It follows that the expression $\lambda_K(x) - \lambda_{K/F}(x)$ is precisely the number of quadratic idele class characters χ of K with $q(\chi) \leqslant x$ such that the corresponding quadratic extension Q of K is not Galois over F. The normal closure of Q over F is then a field L with $\mathrm{Gal}(L/F) \cong D_4$, and the representation $\rho = \mathrm{ind}_{K/F}\chi$ is the irreducible 2-dimensional representation of $\mathrm{Gal}(L/F)$, unique up to isomorphism. Furthermore, ρ is noncanonically induced from K because $\mathrm{Gal}(L/K) \cong (\mathbb{Z}/2\mathbb{Z})^2$.

The map $\chi \mapsto \rho = \mathrm{ind}_{K/F}\chi$ is two-to-one by Frobenius reciprocity, because the two characters occurring in the restriction of ρ to $\mathrm{Gal}(L/K)$, say χ and χ', both induce ρ. The number of such unordered pairs $\{\chi, \chi'\}$ with $q(\chi) = q(\chi') \leqslant x$ is $\lambda(x)$. But $q(\rho) = d_{K/F}q(\chi)$, so $\delta^{\mathrm{non}}_{K/F}(x) = \lambda(x/d_{K/F})$.

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