Mirror Symmetry, SYZ & Enumerative Geometry

$X = \{ f(x_0, \cdots, x_5) = 0 \} \subseteq \mathbb{P}^4_{(x_0,\cdots,x_4)}$, $\deg f = 5$

Calabi–Yau 3-folds $K_X \cong \mathcal{O}_X$

n_d = # of rational curves of degree d in X

= # of homog. poly. $x_i(s,t)$ of degree d

s.t $f(x_0(s,t), \cdots, x_4(s,t)) \equiv 0$.

This is a classical problem in enumerative geometry.

$n_1 = 2875$, $n_2 = 609250$, $n_3 = ??$

(Candelas–de la Ossa–Greene–Plesser '90)

generating function of n_d $\Longleftarrow$ period integral (residue calculation)

+ ODE

on another $\underline{\text{CY 3-fold } \check{X}}$

mirror of $\check{X}$

Q: How to find the mirror $\check{X}$?

(Strominger–Yau–Zaslow '96)

$$\begin{tikzcd}
L \subseteq X \arrow[r, leftrightarrow, "\text{mirror}"'] \arrow[d, "\exists \pi"'] & \check{X} \supseteq \check{L} \arrow[d, "\check{\pi}"] \\
B \arrow[r, equal] & B
\end{tikzcd}$$

① Lagrangian $\omega|_L = 0$

Special $\mathrm{Im}\,\Omega|_L = 0$

dual torus

② mirror

near large complex structure limit (LCSL)

③ Complex structure of $\check{X}$ received "quantum correction" from holo. discs w/ boundary of π.

There might exist singular fibres

$$\begin{tikzcd} X \supseteq \left(T^*B_0/\Lambda, \omega\right) \arrow[d] \arrow[rr, bend left, "\text{mirror}"] & & \left(TB_0/\Lambda^*, J_0\right) \arrow[d] \arrow[r, squiggly, "\text{quantum correction}"'] & (\check{X}, J) \\ B_0 \arrow[rr, equal] & & B & \end{tikzcd}$$

compactification

fibres $\quad L = H_1(L_b, \mathbb{R}) / H_1(L_b, \mathbb{Z}) \quad \overset{V}{\Lambda}$

$$\check{L} = H^1(L, \mathbb{R}) / H^1(L, \mathbb{Z}) \quad \overset{V^*}{\Lambda^*}$$

$$= \mathrm{Hom}(\pi_1(L), U(1))$$

∇ = flat $U(1)$-connection on L

before "quantum correction" $\quad z_A = \exp\left(-\int_A \omega\right) \mathrm{hol}_\nabla(\partial A)$

$A \in H_2(X, L)$

Notice the implicit dependence on the symplectic form

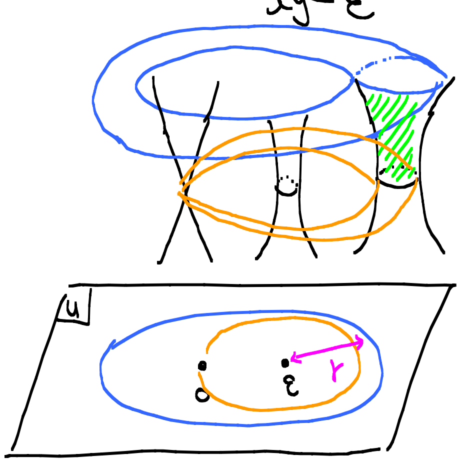
Q: How do we get the "quantum corrected" mirror $\check{X}$?

ex. (Gross, Auroux) $\quad X = \mathbb{C}^2 \setminus \{xy = \varepsilon\}$

$$\omega = \frac{i}{2}\left(dx \wedge d\bar{x} + dy \wedge d\bar{y}\right)$$

$$\Omega = \frac{dx \wedge dy}{xy - \varepsilon}$$

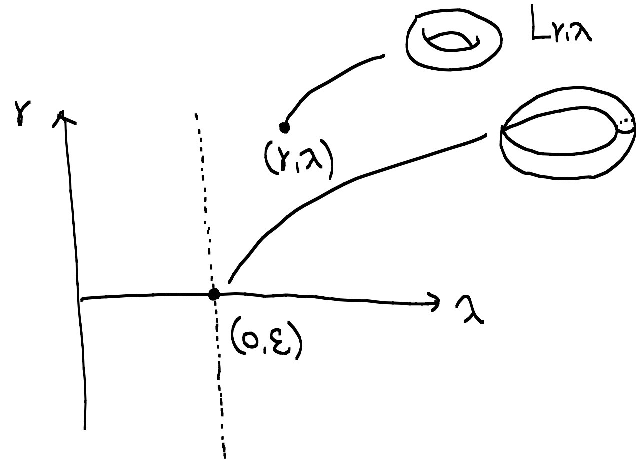
$$\begin{tikzcd} \mathbb{C}^2_{(x,y)} \arrow[d, "p"'] & (x,y) \arrow[d, mapsto] \\ \mathbb{C}_u & xy \end{tikzcd}$$



$$\begin{tikzcd} X \arrow[d, "\pi"'] & (x,y) \arrow[d, mapsto] \\ \mathbb{R}^2 & \left(|x|^2 - |y|^2, |xy - \varepsilon|\right) \end{tikzcd}$$

λ: symplect area $\quad r$

$\int \omega$



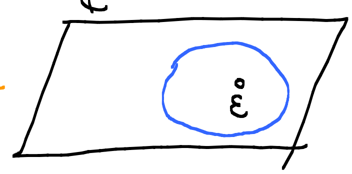
Q: What are the holo. discs w/ boundary on $L_{r,\lambda}$

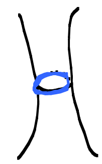
$$(D^2, \partial D^2) \xrightarrow{f} (X, L_{r,\lambda})$$

$p \circ f \searrow \quad \downarrow p$

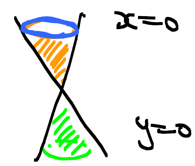
$p\circ f = \text{Const}$ (maximal principle)

$\mathbb{C}^2$



- $p \circ f = u \neq 0$, $p^{-1}(u) =$ 

$\Rightarrow f = \text{const}$ by maximal principle

- $p \circ f = 0$, $p^{-1}(0) =$ 

$\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$ ① $z_1 = z_1'$, $z_2 = z_2'(1+z_1'^{-1})^{-1}$ $\frac{dz_1'}{z_1'} \wedge \frac{dz_2'}{z_2'}$

composition of ①②③ = id

$\Rightarrow$ well-defined gluing

z_1, z_2 $\qquad$ z_1', z_2' $\quad \rightsquigarrow \quad \check{X} = \{ uv = 1 + z_1 \}$

another CY

branch cut

③ $z_1 \mapsto z_1$, $z_2 \mapsto z_2 z_1^{-1}$

Picard-Lefschetz formula

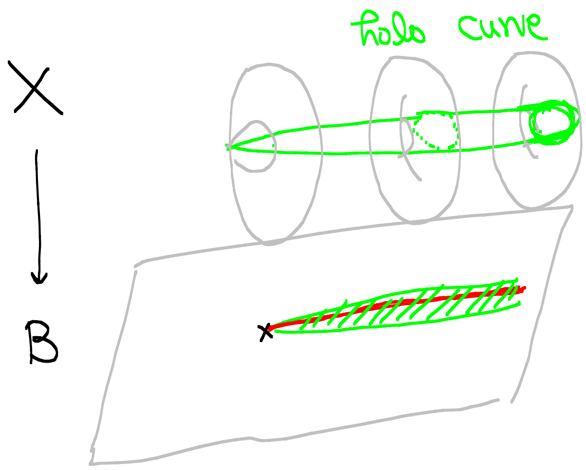
② $z_1' = z_1$

$z_2' = z_2 \underbrace{(1+z_1)}$

from counting of holo. discs

Siu-Cheung generalized the mirror construction to toric Calabi-Yau manifolds.

LCSL & Enumerate Geometry



LCSL "=" collapsing of SYZ fibration

- Projection of holo. curves converge to $\underline{\text{1-skeletons}}$ at $\underline{\text{LCSL}}$
  tropical curves — certain adiabatic limit in geometric analysis
- $\underline{\text{Suitably defined holo. curve counting}}$ is deformation invariant. — Gromov–Witten invariants

$\Longrightarrow$ Enumeration of 1-skeleton recovers the enumeration of holo. curves.

ex.

$$\begin{tikzcd} X_\Delta \arrow[d, "\text{moment map}"'] & \supseteq & (\mathbb{C}^*)^n \arrow[d, "\mathrm{Log}"] & \ni & (X,Y) \arrow[d, mapsto] \\ \Delta & \supseteq \Delta^\circ \cong & \mathbb{R}^n & \ni & (\underset{x}{\log|X|}, \underset{y}{\log|Y|}) \end{tikzcd}$$

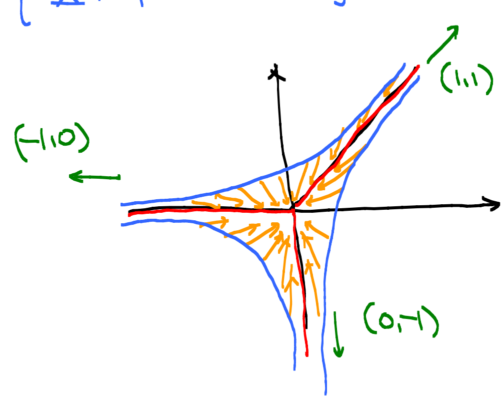
Legendre transform

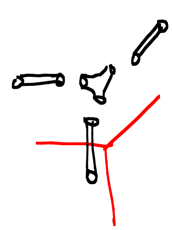
Mikhalkin $\circlearrowright \; H_t(X,Y) = \left(|X|^{\frac{1}{\log t}} \frac{X}{|X|}, \; |Y|^{\frac{1}{\log t}} \frac{Y}{|Y|} \right)$

as $t \to \infty$, $\omega|_L \to 0$

What happens to holo. Curves as $t \to \infty$?

ex. $\{X + Y + 1 = 0\} \subseteq (\mathbb{C}^*)^2$





corner locus of $\max\{x, y, 1\}$

$= V(x \oplus y \oplus 1)$

tropical curve

$1 \cdot (1,1) + 1 \cdot (0,-1) + 1 \cdot (-1,0) = 0$

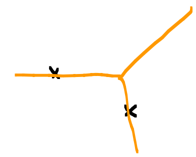
balancing condition

Moral: LCSL $\rightsquigarrow$ Tropical geometry

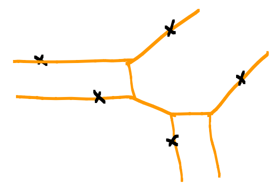
toric degeneration

- Bezout theorem $\Longleftrightarrow$ Pick's theorem of lattice points.

- Two points determines a line.



- Five points determines a conic.



(Mikhalkin, Nishinou–Siebert) Weighted count of tropical curves

= enumeration of holo. curves

log Gromov–Witten invariants

(Collins–Jacob–L. '19) $X = \mathbb{P}^2 \setminus E$, E: smooth elliptic curve

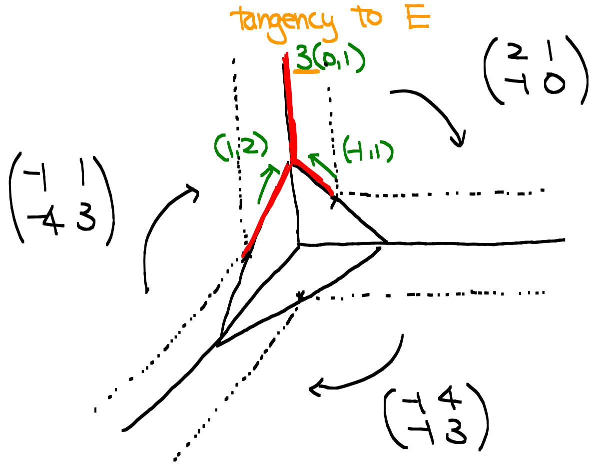
Conjecture of Auroux '07

$$\exists \downarrow \quad \text{SYZ fibration}$$

$$B \underset{\text{top}}{\cong} \mathbb{R}^2$$

(Lau–Lee–L. '20) As an affine manifold,

$B =$

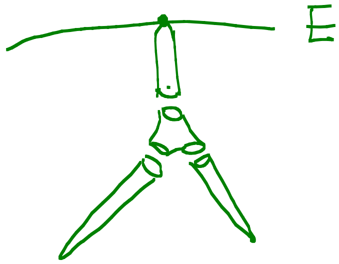


$\left|\det\begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}\right| = 3$

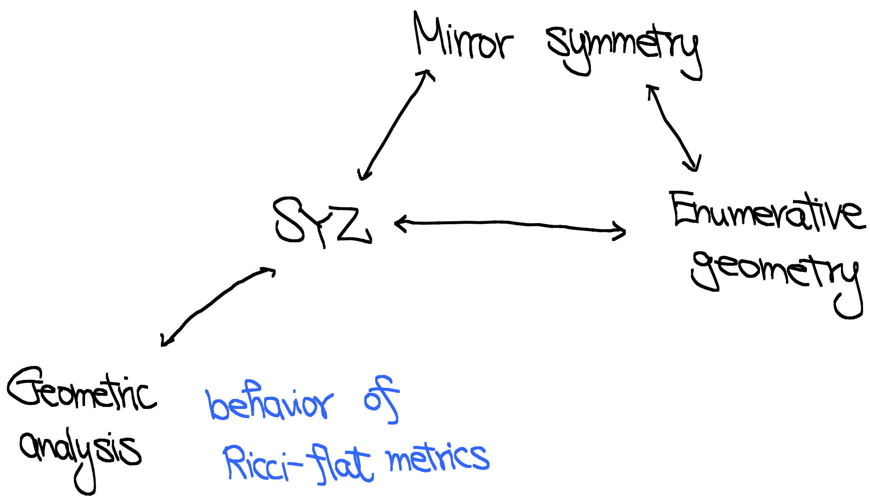
There are 3×3 rational curves intersecting E at 1 point.

They are simply the tangent lines of the 3-torsion points of E!

(L. '20) Enumeration of $\mathbb{A}^1$–curves



To sum up,



Thank You