New Construction of Special Lagrangian Fibrations

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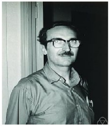
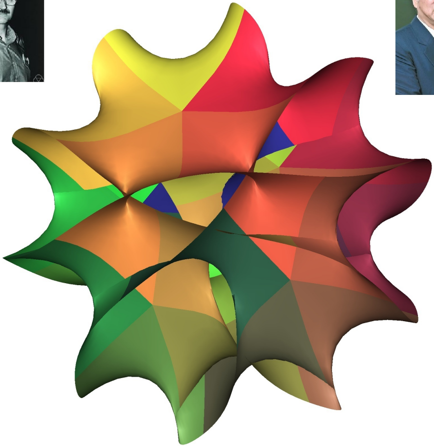
Outline of the Talk

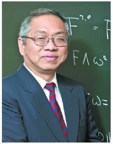
- Calabi-Yau Manifolds and Strominger-Yau-Zaslow Conjecture
- Main Theorems and Applications
- Sketch of the Proofs

Calabi-Yau Manifolds

- Calabi-Yau manifold.







Calabi-Yau Manifolds

- Calabi-Yau n-fold X
  =higher dimension analogue of elliptic curves.
  =complex manifold X with
  1. nowhere vanishing holomorphic n-form Ω
  2. d-closed non-degenerate positive $(1,1)$-form ω such that $\omega^n = c\Omega \wedge \bar{\Omega}$.

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  2. d-closed non-degenerate positive $(1,1)$-form ω such that $\omega^n = c\Omega \wedge \bar{\Omega}$.
- Examples:
  - degree 5 hypersurface in $\mathbb{P}^4$ (quintic 3-fold).
  - (Tian-Yau) complement of a smooth anti-canonical divisor in a Fano manifold

Mirror Symmetry

Physicists found out that each Calabi-Yau manifold X admits a mysterious partner $\check{X}$ such that

1. $A(X) = B(\check{X}), A(\check{X}) = B(X)$,
2. where A/B denote some invariants of symplectic/complex geometry.

The Calabi-Yau manifold $\check{X}$ is called the mirror of X.

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Question

How do we find the mirror of X?

Strominger-Yau-Zaslow Conjecture

Conjecture (Strominger-Yau-Zaslow '96)

- *A Calabi-Yau X admits special Lagrangian torus fibration.*
- *The mirror $\check{X}$ is the dual torus fibration.*
- *The Ricci-flat metric of $\check{X}$ need instanton correction from holomorphic discs with boundary on the torus fibres.*

- $L \subseteq X$ is special Lagrangian if $\omega|_L = 0$, $\Omega|_L = \mathrm{vol}|_L$.
- We know the existence of the Ricci-flat metric for 40 years but don't know much of the description of it.

The three problems form an iron triangle:

$$\begin{tikzcd}
 & \text{Ricci-flat metric} \arrow[dr] & \\
\text{SYZ fibration} \arrow[ur] & & \text{instanton correction/holo. disc} \arrow[ll]
\end{tikzcd}$$

Existing Examples

- Almost special Lagrangian fibration (not wrt Ricci-flat metric)
  1. (Gross '00) toric Calabi-Yau manifolds
  2. (Goldstein '02) Borisov-Voison Calabi-Yau 3-folds
- True special Lagrangian fibration
  1. Complex tori with flat metric or
  2. hyperKähler rotation of holomorphic Lagrangian fibration in a hyperKähler manifold.

Ricci-Flat Metrics on Log Calabi–Yau Manifolds

- Y = projective manifold of dimension n with
- $D = s^{-1}(0)$ smooth effective anticanonical divisor, where $s \in H^0(Y, -K_Y)$ and no curves disjoint from D can be realized as linear combination of curves support in D.

Then $Y \setminus D$ admits a non-vanishing holomorphic volume form Ω

Theorem (Tian–Yau '90)

There exists a complete Ricci-flat metric ω_{TY} on $X = Y \setminus D$ with asymptotics near D

$$\omega_{TY} \sim \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}(-\log \| s \|)^{(n+1)/n}.$$

Question: Is there a SYZ fibration on X?

Main Result 1: New Special Lagrangian Fibrations

Theorem 1 (Collins-Jacob-L. '19)

Y=weak del Pezzo surface, $D \in |-K_Y|$ smooth.
Then $X = Y \backslash D$ admits a special Lagrangian fibration with a special Lagrangian section with respect to the Tian-Yau metric.

- This solves conjectures of Yau and Auroux '08.
- Probably the only non-trivial example so far.
- (Collins-Jacob-L.) Y=rational elliptic surface, D=I_d type singular fibre.

Main Result 2: Application to Mirror Symmetry

Theorem 2 (Collins-Jacob-L. '19)

Let $\check{X}$ be a suitable hypKähler rotation of X. Then $\check{X}$ is the fibrewise compactification of the Landau-Ginburg mirror of X.

- HyperKähler rotation gives the mirror!
- $\check{X}$ can be compactified to be a rational elliptic surface of with an I_d fibre adding at infinity, $d = (-K_Y)^2$.
- (Auroux-Kartzarkov-Orlov '05) showed that the above is the compactification of the Landau-Ginzburg mirror of X.
- We don't have a Floer theoretic explanation of this phenomenon yet.

Relation with Gross-Siebert Program

- SYZ conjecture is served as a guiding principle for mirror symmetry. A lot of implications are proved.
- To avoid the analysis difficulties, Kontsevich-Soibelman, Gross-Siebert developed the algebraic alternative for SYZ dual fibration constructing the mirror.

Theorem (Lau-Lee-L.)

The complex affine structure of the SYZ fibration of $\mathbb{P}^2 \backslash E$ coincides with the one of Gross-Siebert program.

This lays out the foundation of the comparison of family Floer mirror with the mirror constructed in Gross-Siebert program.

Why SYZ Fibrations?

(Hitchin) $\exists$ integral affine structure on B_0 with integral affine coordinates

$f_i(u) = \int_{\gamma_i} \mathrm{Im}(e^{-i\vartheta}\Omega)$

Lemma

If there exists a family of special Lagrangian torus L_t bounding holomorphic discs in relative class $\gamma \in H_2(X, L_t)$ in X, the L_t sit over an affine line.

Better control of locus of Lagrangian fibres bounding holomorphic discs if the fibration is special.

Application of SYZ Fibrations in Enumerative Geometry

Usually, it is hard to compute family Floer mirror explictly.

Theorem (Cheung-L.)

An explicit calculation of family Floer mirror for certain HK surface.

For geometric interpretation of the slab functions in GS programs:

Theorem (L.-)

Equivalence of open Gromov-Witten invariants and weighted count of tropical discs counting for HK surfaces with SYZ fibration.

Similar spirit is used to proved

Theorem (Hong-L.-Zhao '18)

Tropical/holomorphic correspondence for discs with interior bulk insertions and quantum period theorem for toric Fano surfaces.

General Existence Theorem of SYZ Fibration

Theorem 3 (Collins-Jacob-L.- '19)

X=complete hyperKḧler surface with

1. *bounded sectional curvature and injectivity radius decay mildly*
2. $\chi(X) < \infty$
3. *$L \subseteq X$ smooth or immersed special Lagrangian torus with $[L]^2 = 0$ and $[L] \in H_2(X, \mathbb{Z})$ primitive.*

Then X admits a special Lagrangian fibration with L a smooth fibre. Moreover, the singular fibres are those classified by Kodaira.

So existence of SYZ fibration for a HK surface reduce to the existence of a single special Lagrangian torus.

Remarks on the new Existence Theorem

- The theorem is automatic from Riemann-Roch theorem when X is a K3 surface.
- Minimal use of the hyperKähler condition.
- The main advantage of the theorem is the existence of special Lagrangian fibration in a (log) Calabi-Yau surface with explicit equation.

Existence of Special Lagrangian Submanifolds

Theorem 4 (Collins-Jacob-L.- '19)

Y=Fano manifold with $D \in |-K_Y|$ smooth $L \subseteq D$=special Lagrangian submanifold then $X = Y \backslash D$ contains a special Lagrangian submanifold with topology the same as $L \times S^1$.

- This produces a lot of new examples of special Lagrangians in log Calabi-Yau manifolds. For instance,

Theorem

Every log Calabi-Yau 3-folds contains infinitely many special Lagrangian tori.

- Theorem 3 + Theorem 4 $\Rightarrow$ Theorem 1.

Proof of Theorem 4

- Ansatz Special Lagrangians
- Lagrangian Mean Curvature Flow

Geometric Input

- Y=RES, then ω_{TY} is asymptotically cylindrical relatively easy.
- Y=del Pezzo surface,
  then first eigenvalue λ_1 and injectivity radius are degenerating.
- Need qualitative version of all estimates.

Calabi Ansatz

- D=projective Calabi-Yau of dimension $n-1$.
- $X_{\mathcal{C}}$=neighborhood of zero section of $N_{D/Y}$, $\pi : X_{\mathcal{C}} \to D$.
- $\Omega_{\mathbb{C}} = \frac{f(z)}{w} dz_1 \wedge \cdots \wedge dz_{n-1} \wedge dw$.
- $\omega_{\mathbb{C}} = \sqrt{-1}\partial\bar{\partial}\frac{n}{n+1}(-\log|\xi|_h^2)^{\frac{n+1}{n}}$, where $\omega_D = \sqrt{-1}\partial\bar{\partial}h$.
- Set $l_0 = \big(-\log|\xi|_h^2\big)^{\frac{1}{2n}}$. Then
  1. $|\nabla^k Rm| \le C_k l_0^{-(k+2)}$ has good control and
  2. $C_\iota^{-1} l_0^{1-n} \le \text{inj} \le C_\iota l_0^{1-n}$ degenerates.

Ansatz Special Lagrangian Tori

- L special Lagrangian in D.
- $L_\epsilon = \pi^{-1}(L) \cap \{|\xi|_h^2 = \epsilon\}$ topologically $L \times S^1$.
- $\Omega_{\mathcal{C}}|_{L_\epsilon} = \sqrt{-1}\pi^*\Omega_D \wedge d\theta|_{L_\epsilon}$

  $\omega_{\mathcal{C}} = \frac{\sqrt{-1}}{n}(-\log|\xi|_h^2)^{\frac{1}{n}-1}\partial l_0 \wedge \bar{\partial} l_0 + (-\log|\xi|_h^2)^{\frac{1}{n}}\pi^*\omega_D$.
- In particular, L_ϵ is a special Lagrangian wrt $(\omega_{\mathcal{C}}, \Omega_{\mathcal{C}})$.
- Special Lagrangian fibration D implies special Lagrangian fibration on $X_{\mathcal{C}}$.

Some Geometric Quantities of L_ϵ

- Induced metric $g_{\mathcal{C}}|_{L_\epsilon} = (-\log \epsilon)^{\frac{1}{n}-1}\frac{1}{n}d\theta^2 + (-\log \epsilon)^{\frac{1}{n}}\pi^* g_D|_L$.
- Second fundamental form

$$|A|^2 \leq C(L,n)(-\log \epsilon)^{-\frac{1}{n}}$$

  where $C(L,n)$ depend only on n and the second fundamental form of $L \subseteq (D, g_D)$.
- mean curvature vector $H = 0$
- L_ϵ is κ-non-collapsing at scale r_ϵ for

$$\kappa = \frac{\sqrt{n}\kappa_L}{2^{n-1}} \text{ and } r_\epsilon = \frac{2\pi}{\sqrt{n}}(-\log \epsilon)^{\frac{1-n}{2n}}.$$

First Eigenvalue Estimate of L_ϵ

- Recall the Rayleigh quotient for first eigenvalue

$$\lambda_1 = \inf_{f \in C^\infty(L)} \frac{\int_L |\nabla f|^2 \mathrm{Vol}_L}{\int_L |f|^2 \mathrm{Vol}_L}.$$

- $|d\pi^* f|^2_{g_C} = (-\log(\epsilon))^{-1/n} \pi^* |df|^2_{g_D}$

$$\Rightarrow \lambda_1(L_\epsilon, g_{\mathbb{C}}) \leq \frac{\lambda_1(L, g_D)}{(-\log(\epsilon))^{1/n}}.$$

- (Li-Yau) M^n compact Ricci-flat manifold without boundary. Then $\lambda_1 \geq C/(n-1)d^2$, where $d = \mathrm{diam}(M)$ and $C = C(n)$.
- From the expression $g_C|_{L_\epsilon}$ and Li-Yau gives the other direction of inequality.

Notion of Bounded Geoemtry

Definition

L has (C, K, δ')-bounded geometry if

- $C^{-1}K < l_0|_L < CK$.
- $|A|^2 \le CK^{-2}$.
- $|H|^2 \le Ce^{-\delta' K^{2n}}$.
- $C^{-1} \le vol_L \le C$.
- $C^{-1}K^{-2} \le \lambda_1 \le CK^{-2}$.
- *$g_{TY}|L$ is κ_0-non-collapsing on scall r_0 with $\kappa_0 \ge C^{-1}$, $r_0 \le C^{-1}K^{1-n}$.*

Lemma

L_ϵ has (C, K, δ')-bounded geometry, with $K = (-\log \epsilon)^{\frac{1}{n}}$.

Moser's Trick

Lemma (HSVZ '18)

$\exists$ compact set $K \subseteq X, K_{\mathcal{C}} \subseteq X_{\mathcal{C}}$ and diffeomorphism $\Phi : X_{\mathcal{C}} \backslash K_{\mathcal{C}} \to X \backslash K$ such that

$$|\nabla_{g_{\mathcal{C}}}(\Phi^* \omega_{TY} - \omega_{\mathcal{C}})|_{g_{\mathcal{C}}} = O(e^{-\delta l_0^{2n}}),$$

for some $\delta > 0$. Same for J, Ω, g.

- $\omega_{TY} - (\Phi^{-1})^* \omega_{\mathcal{C}} = d\beta$
- Run Moser's trick with the geometric quantity controls are preserved. L_ϵ is an almost special Lagrangian wrt ω_{TY} and has (C, K, δ')-bounded geometry.

Lagrangian Mean Curvature Flow

- Let L be a graded Lagrangian submanifold in X. Then the phase $\theta : L \to \mathbb{R}$ is the function such that

$$\Omega|_L = e^{i\theta} vol_L.$$

  L is a special Lagrangian if θ is a constant.
- The mean curvature $\vec{H} = J\nabla\theta$ and the mean curvature flow is given by evolving family of immersions $F_t : L \to X$ with

$$\frac{\partial}{\partial t} F_t = \vec{H}.$$

- (Smoczyk) Maslov zero Lagrangian condition is preserved under mean curvature flow in Kähler–Einstein manifolds.

Lagrangian Mean Curavature Flow continued

- Lagrangian mean curvature flow also preserves the Hamiltonian isotopy class.
- (Thomas–Yau '01) There exists at most one special Lagrangians in a given Hamiltonian isotopy class. Thomas–Yau proposed to use mean curvature flow to find the unique special Lagrangian representative analogue of stable bundles in the B-side.
- (Neves '07) Example of Lagrangian mean curvature flow can develop finite time singularities in dimension four.
- (Joyce '14) Proposed to use Lagrangian mean curvature flow to study the Bridgeland stability conditions on Fukaya categories.

Some Notations for LMCF

$F : L \to X$ Lagrangian immersion w/ $(x_1, \cdots, x_n)$ local coor. on L.

- $F_i = \frac{\partial F}{\partial x_i}$ and $\nu_i = JF_i$.
- $g_{ij} = \langle F_i, F_j \rangle$.
- $h_{ijk} = -\langle \nu_i, \nabla_{F_j} F_k \rangle$.
- $\alpha_H = H_i dx^i$, where $H_i := g^{kl} h_{ikl}$.
- $|A|^2 = g^{ij} g^{pq} g^{lm} h_{ipl} h_{jqm}$.

Evolution Equations of LMCF

- $\frac{\partial}{\partial t}\theta = \Delta\theta$.
- $\frac{\partial}{\partial t}\alpha_H = dd^*\alpha_H$. In particular, α_H stay in the same cohomology class. Zero Maslov index condition is preserved.
- $\frac{\partial}{\partial t}\mathrm{vol}_L = -|H|^2\mathrm{vol}_L$.
- $\frac{\partial}{\partial t}H_j = \nabla_j\nabla^i H_i$.

L_2-Estimates of the Mean Curvature

Lemma (Li '09)

Assume along the flow $|A(t)|^2 \leq \Lambda$, $|H(t)|^2 \leq \epsilon$ and $\lambda_1 \geq \delta$. Then

$$\frac{\partial}{\partial t} \int_{L_t} |H|^2 Vol \leq -(\delta - 2\Lambda\epsilon) \int_{L_t} |H|^2 Vol.$$

- As a corollary, $\int_{L_t} |H|^2 \mathrm{vol} \searrow 0$ exponentially.
- $\frac{\partial}{\partial t} \int_{L_t} |H|^2 \mathrm{vol} \leq 2 \int g^{jl} H_l \nabla_j \nabla^i H_i + 2|A||H|^3 - |H|^4 \mathrm{vol}.$
- $\frac{\partial}{\partial t} \int_{L_t} |H|^2 \mathrm{vol} \leq -2 \int_{L_t} |\nabla^i H_i|^2 + 2 \sup_{L_t} |A||H| \int_{L_t} |H|^2 \mathrm{vol}.$

Let f_i are the eigenvector of Δ with eigenvalue λ_i.

$$\begin{aligned}\int |\nabla^i H_i|^2 = \int |\Delta\theta|^2 \geq \sum_i \lambda_i^2 \int f_i^2 \\ \geq \lambda_1 \sum_i \lambda_i \int f_i^2 = -\lambda_1 \int \theta\Delta\theta = -\lambda_1 \int |\nabla\theta|^2.\end{aligned}$$

From L_2 to Pointwise Estimates

Lemma

(M, g) Riemannian, S tensor with

1. *(L^2-estimates)* $\int_M |S|^2 \leq \epsilon$
2. *(gradient estimate)* $|\nabla S| \leq C$
3. *(non-collapsing)* $\int_{B_r} |S| \geq \min|S| \cdot \kappa_0 r^n$, *for* $r < r_0$.

Then $\sup_M |S| \leq \left(\frac{1}{\kappa_0} + C\right)\epsilon^{\frac{1}{n+2}}$, *if* $\epsilon < r_0^{n+2}$.

Conservation of Pointwise Estimates

- $\frac{\partial}{\partial t}|A|^2 \le \Delta|A| + 8|A|^2 + 20|Rm||A| + 4|\nabla Rm|$.
- $\Lambda := \sup_{L_0} |A(0)|$, then from comparison principle $|A(t)| \le \Lambda + C_A(l_0)t$. A priori can be out of control
- $\frac{\partial}{\partial t}|H| \le \Delta|H| + 2|A|^2|H| + |Rm||H|$.
- Again from comparison principle, $|H(t)| \le |H(0)|e^{C_H(l_0)t}$.

First Eigenvlaue Estimates under LMCF

- Recall the Rayleigh quotient for first eigenvalue

$$\lambda_1(t) = \inf_{f \in C^\infty(L)} \frac{\int_L |\nabla f|^2_{g_t} \mathrm{Vol}_t}{\int_L |f_t|^2 \mathrm{Vol}_t}.$$

- Set $f_t = f - \frac{1}{\mathrm{vol}_{L_t}} \int_L f \mathrm{vol}_t$.
- $|\frac{\partial}{\partial t} \int_L |\nabla f|^2_{g_t} \mathrm{vol}_t| \leq 2 \sup_L |\partial_t g|_{g_t} \big(\int_L |\nabla f_t|^2_{g_t} \mathrm{vol}_t \big)$.
- $|\frac{\partial}{\partial t} \int_L f_t^2 \mathrm{vol}_t| \leq \sup_L |\partial_t g_t|_{g_t} \int_L f_t^2 \mathrm{vol}_t$.
- Set $\mu(t) = \int_0^t \sup_L |\partial_s g_s|_{g_s} ds$, then by comparison principle

$$e^{-3\mu(t)} \lambda_1(0) \leq \lambda_1(t) \leq e^{3\mu(t)} \lambda_1(0).$$

Non-Collapsing Estimates

Lemma

$B_{g_0}(x,r) \subseteq L$ geodesic ball with $Vol(B_{g_0}(x,r)) \geq \kappa r^n$, $r < R$.
$\Rightarrow Vol(B_{g_t}(x,r)) \geq \kappa' r^n$

- $\frac{\partial}{\partial t}\eta_{ij} = -2a_{ij} = -2H^p h_{pkl}.$
- $E(t) := \int_0^t (\sup_{L_t} |A|. \sup_{L_u} |H|) ds$

  $\Rightarrow e^{-E(t)} d_{g_0}(p,q) \leq d_{g_t}(p,q) \leq e^{-E(t)} d_{g_0}(p,q)$ and

  $\mathrm{Vol}_{L_t} \geq e^{-E(t)} \mathrm{Vol}_{L_0}.$
- $\int_{B_{g_t}(x,r)} \mathrm{Vol}_{L_t} \geq \int_{B_{g_0}(x', e^{-E(t)} r)} e^{-E(t)} \mathrm{Vol}_{L_0} \geq \kappa e^{-(n+1)E(t)} r^n.$

Bounded Geometry under LMCF

Putting together the estimates for second fundamental form, mean curvature, first eigenvalue and non-collapsing constants under LMCF:

Lemma

L=Maslov zero Lagrangian w/ (C, K, δ')-bounded geometry. Then LMCF of L preserves $((1+\delta)C, K, \delta')$-bounded geometry for $t \in [0, \alpha K^2)$, $\alpha = \alpha(C, \delta)$.

Smoothing Estimates

From estimates of second fundamental form to its higher derivatives:

Lemma

Assume that L_ϵ has (C, K, δ')-bounded geometry for $t \in [0, \alpha K^2)$. Then

$$|\nabla^l A|^2 \leq C(l)\frac{K^{-2}}{t^l}.$$

- We will only need the case $l = 1, 2, 3$ for estimates of $|\nabla^l H|$.

Improved Estimates for A under LMCF

- (Smoczyk) $\frac{\partial}{\partial t}|A|^2 \leq 100(|A||\nabla^2 H| + |A|^3|H| + |Rm||H|)$.
- To estimate $|\nabla^2 H|$,
  $\int |\nabla^2 H|^2 \leq \int |H||\nabla^4 H| \leq \sqrt{C_4} K^{-1} e^{-\frac{\delta'}{2(n+2)^2} K^{2n} - \frac{a}{2(n+2)} t}$.
- Together with the estimate $|\nabla^3 H|$,
  one has $|\nabla^2 H| \sim O\left(e^{-\frac{\delta'}{2(n+2)} K^{2n} - \frac{a}{2(n+2)} t}\right)$.
- Feedback to the top inequality, one has
  $\frac{\partial}{\partial t}|A|^2 \sim O\left(e^{-\frac{\delta'}{2(n+2)} K^{2n} - \frac{a}{2(n+2)} t}\right)$.
- Integrating t, one has $|A(t)|^2 \leq 2CK^{-2} + O(e^{-\frac{\delta'}{2(n+2)} K^{2n}})$.

Lemma

For $K \gg 0$, then the LMCF of L_ϵ with (C, K, δ')-bounded geometry converges smoothly with $(4C, K, \frac{\delta'}{n+2})$-bounded geometry.

- Assume that T is the maximal time of LMCF preserving $(4C, K, \frac{\delta'}{n+2})$-bounded geometry.
- Claim: L_t is of $(3C, K, \frac{\delta'}{n+2})$-bounded geometry.
  This contradicts to the definition ot T.

The proof of Theorem 3

- HyperKähler rotation
- Theory of J-holomorphic curves

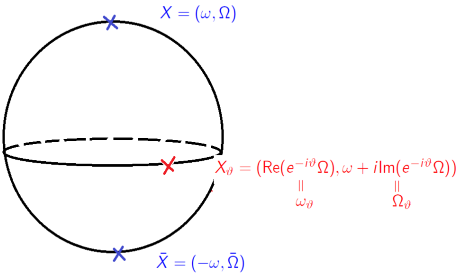
HyperKähler Rotation

- Since $SU(2) \cong Sp(1)$, Calabi-Yau 2-folds are hyperKähler.
- L: holomorphic curve in $X \Leftrightarrow \Omega|_L = 0$

$$\Leftrightarrow \mathrm{Re}(e^{i\vartheta}\Omega)|_L = 0 = \mathrm{Im}(e^{i\vartheta}\Omega_\vartheta)|_L$$

$$\Leftrightarrow \omega_\vartheta|_L = 0 = \mathrm{Im}\Omega_\vartheta|_L$$

- L: special Lagrangian in X_ϑ, $\forall \vartheta \in S^1$.



It suffices to prove the holomorphic version of Theorem 3.

Sacks-Ulenbeck-Gromov compactness

Lemma (Sacks-Ulenbeck,Gromov, Sikrov, Groman, CJL,...)

(X, ω, J): *complete, bounded sectional curvature, injectivity radius decay mildly and J uniformly tamed*
$\Rightarrow$ *The set of J-holo. curves passing through a fixed compact subset K and bounded area is compact.*

- Use monotonicity of J-holomorphic curves to bound the diameters of all such J-holomorphic curves.
- If $\| df_i \|$ bounded, then f_i converges uniformly by Arzela-Ascoli theorem and the limit is holomorphic.
- Otherwise, $\exists x_i$ s.t. $\| df(x_i) \| \to \infty$. There will be a sphere bubble near $x = \lim x_i$.

Monotonicity of J-Holomorphic Curves

J-holomorphic curves follow the monotonicity.

Lemma

With above assumptions, $\exists r_0, C > 0$ *such that for* $r < r_0$
$f : \Sigma \to B(x,r)$ *and* $f(\partial\Sigma) \subseteq \partial B(x,r) \Rightarrow Area(f(\Sigma)) \geq Cr^2$

Monotonicity implies the following diameter estimate.

Lemma

With above assumptions, $\exists R = R(X, K, A)$ *such that*
$C \cap K \neq \emptyset$, $\partial K \subseteq K$ *and* $Area(C) \leq A \Rightarrow diam(C) \leq R$.

Step 1:

- (McLean) Deformation of special Lagrangians are unobstructed.
- $\check{X}$=suitable HK rotation of X.
- $\exists$ tubular neighborhood $U \subseteq \check{X}$ of C with $U \to \mathbb{C}$ elliptic fibration. Fibres are smooth elliptic curves deformable to C.
- $X_1 :=$ union of smooth elliiptic curves deformable to C. Then X_1 is open.
- $X_2 := \partial X_1$, then $X_1 \cup X_2$ is closed in $\check{X}$

  Claim: $X_1 \cup X_2$ is open in $\check{X}$.
- From compactness theorem, X_2 consists of singular elliptic curves deformable to C.

Step 2: Classification of Singular Fibres

Assume that $C_i \to C_0$, where C_0 with components $C_0^{(k)}$ with multiplicity n_k.

- $[C_0]^2 = 0$ and $[C_0^{(k)}].[C_0] = 0$
  then $Q \leq 0$ and $\mathrm{Ker} Q = \mathbb{Z}[C_0]$.
- Claim: the singular fibres are those classified by Kodaira.

- (Adjunction Formula, McDuff, M-White)

$$2\delta + \chi(C_0^{(k)}) = [C^{(k)}]^2 + \deg(f^* T\check{X})$$

implies that $g(C^{(k)}) \leq 1$.

- $g(C^{(k)}) = 1$, then only one component and embedding (I_0).
- $g(C^{(k)}) = 0$, then either
  1. $[C^{(k)}]^2 = 0$, only one components and C_0 is a nodal rational curve (I_1) or a cuspidal curve (II).
  2. $[C^k]^2 < 0$, then $[C_0^{(k)}] = -2$ for all k.
     $[C_0^k].[C^{(k')}] = 2 \Rightarrow$ exactly two components (I_2, III)
     $[C_0^{(k)}].[C_0^{(k')}] = 0 \Rightarrow$ configurations are affine Dynkin diagrams
     $\tilde{A}_n$ (I_n, IV), $\tilde{D}_n$ (I_n^*), $\tilde{E}_6$ (IV^*), $\tilde{E}_7$ (III^*), $\tilde{E}_8$ (II^*).

Step 3: $X_1 \cup X_2 = \check{X}$

Claim: $X_1 \backslash X_2$ is path connected.

- Singular curves of the 2nd kind is rigid.
- Singular curves of the 1st kind is rigid.
- Thus, only singular curves of the 1st kind can converges to the singular curves of the 2nd kind
- This can't happen since $\chi(\check{X}) = \chi(X) < \infty$.

Step 4: Fibration Structures

- $B := \check{X}/\sim$, where $x \sim y$ if there exists an elliptic curve contains both x, y.
- (Hithin, Voisin) complex structure near $b \in B_0$ corresponding to a smooth elliptic curve.
- Extend the complex structure to b in discriminant locus.
- $\check{X} \to B$ is continuous, holo. over B_0, locally bounded $\Rightarrow \check{X} \to B$ holo., genus one fibration.

Step 5: More Structures

- $[L] \in H_2(X, \mathbb{Z})$ primitive $\Rightarrow$ no multiple fibre.
- paths in B locally liftable and connected fibres $\Rightarrow \pi_1(X) \twoheadrightarrow \pi_1(B)$
- In particular, $\pi_1(X)$ torsion implies $\pi_1(B)$ torsion.
- B open Riemann surface $\Rightarrow$ deformation retract to CW 1-complex. Thus, B simply connected.
- (uniformization) $B \cong D^2$ or $\mathbb{C}$.
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  Former case, the coordinate function pull back to a non-constant bounded holo. function on $\check{X}$. Yau said NO!

Above all holds for del Pezzo surfaces or rational elliptic surfaces.

Compactification of $\check{X}$

- $D_\infty^* \subseteq B$ neighborhood of ∞.
  $\Rightarrow$ There exists local section of $\check{X} \to B$ over D_∞^*.
  $\rightsquigarrow D_\infty^* \to M_{1,1} \xrightarrow{j} \mathbb{C}$
- $j : D_\infty^* \to \mathbb{C}$ unbounded calculated on the model (Schwartz lemma) $\Rightarrow$ no essential singularity.
- $j : D_\infty^*$ has a pole and extends to $D_\infty \to \bar{M}_{1,1} \xrightarrow{j} \mathbb{P}^1$.
- The monodromy near infinity is $\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$, where $d = (-K_Y)^2$.
- Glue the universal family to $\check{X} \to B$ to get $\check{Y}$.

- From classification of minimal surfaces $\Rightarrow \check{Y}$ is not minimal.
- (canonical bundle formula)

$$K_{\check{Y}} \cong \pi^*(K_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(k)), \quad k \geq 0.$$

- (-1)-curves are sections $\Rightarrow k = 1$.
- (Castulnuvo's criterion for rationality)
  $p_2(\check{Y}) = q(\check{Y}) = 0 \Rightarrow \check{Y}$ is rational.
  canonical bundle formula $\Rightarrow p_2(\check{Y}) = 0$ and
  $h^1(\check{Y}) = 0 \Rightarrow q(\check{Y}) = 0$.
- Thus, $\check{Y}$ is a rational elliptic surface.

THANK YOU!